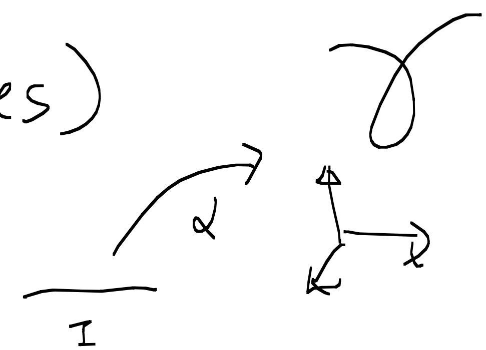
Main Ref : Differential Geometry and Its Applications, 2nd ed. by John Oprea.

Ch1 : Geometry of Curves

1.1 Parametrized Curves. (Parametric Curves)



Def :
- Curve in $\mathbb{R}^3$ is a continuous mapping $\alpha : I \to \mathbb{R}^3$, where I is a an interval.
- $t \in I$ is called a parameter of α.
- $\alpha(t) = (\alpha^1(t), \alpha^2(t), \alpha^3(t))$ is a parametrization of α.

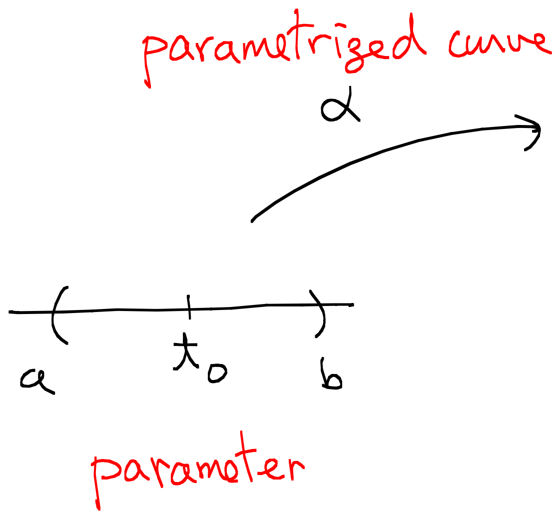
Def: A (parametrized) differentiable (or smooth) curve is a differentiable mapping $\alpha : I \to \mathbb{R}^3$ of an open interval $I = (a, b)$ into $\mathbb{R}^3$.

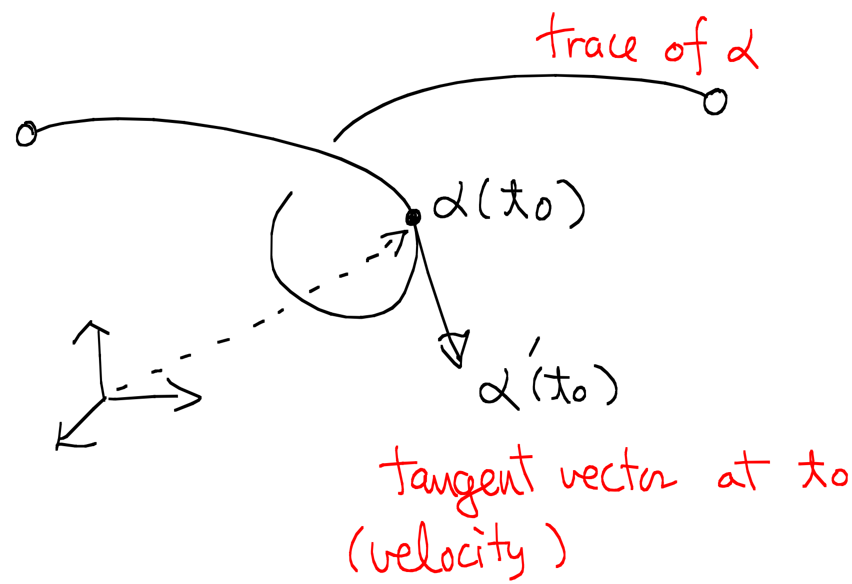
Def: The velocity vector (tangent vector) of α at $t_0 \in I$ is defined to be

$$\alpha'(t_0) = \left(\left.\frac{d\alpha^1}{dt}\right|_{t=t_0}, \left.\frac{d\alpha^2}{dt}\right|_{t=t_0}, \left.\frac{d\alpha^3}{dt}\right|_{t=t_0} \right)$$

(provided α is differentiable at $t = t_0$.)

Remark: $\alpha(I) \subset \mathbb{R}^3$ is called the trace of α. And different parametrized curves may have the same trace.

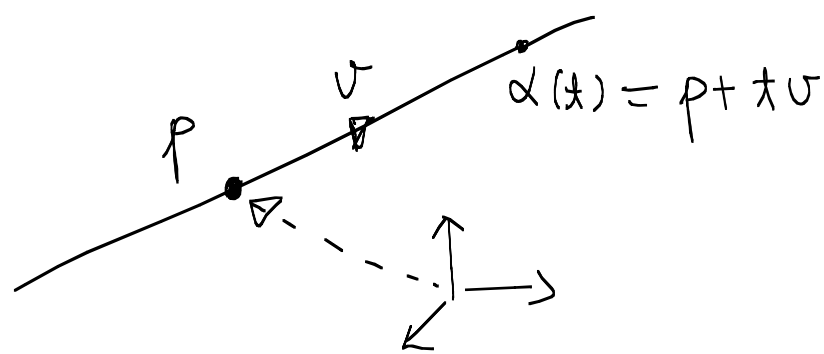




<u>eg 1.1.1</u> (line in $\mathbb{R}^3$)

$\alpha(t) = p + tv$ is a parametrized curve in $\mathbb{R}^3$,

where p & v $(\neq 0)$ are 3-vectors. Note: $I = (-\infty, +\infty)$

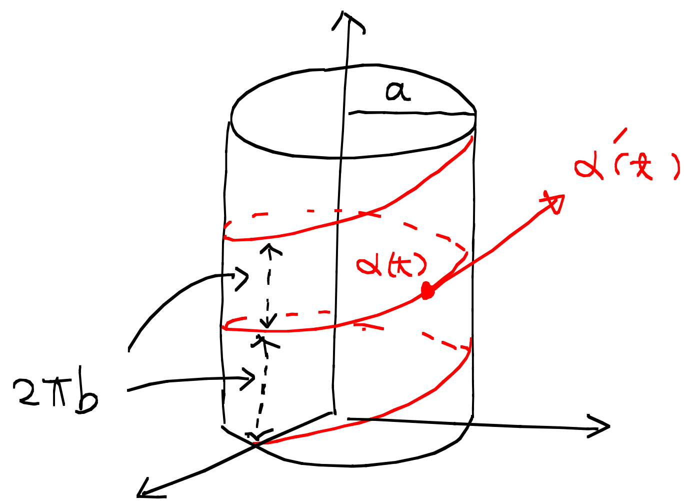


(See Oprea for an example of parametrized line segment.)

eg 1.1.2 (Helix in $\mathbb{R}^3$, eg 1.1.18 in Oprea)

$$\alpha(t) = (a\cos t, a\sin t, bt), \quad t \in (-\infty, +\infty)$$

(a, b are nonzero constants.)



Def:

- $\alpha(t)$ is <u>regular</u> if $\underline{\alpha'(t) \neq 0,\ \forall t \in I}$.

- $|\alpha'(t)|$ is called the <u>speed</u> of α.

<u>eg 1.1.3</u> Both the line and the helix are regular:

line: $\alpha(t) = p + tv \quad (v \neq 0)$

$\Rightarrow \alpha'(t) = v \neq 0$

helix: $\alpha(t) = (a\cos t,\ a\sin t,\ bt),\ (a \neq 0, b \neq 0)$

$\Rightarrow \alpha'(t) = (-a\sin t,\ a\cos t,\ b) \neq 0$

<u>eg 1.1.4</u> (<u>Cusp</u> in a plane, eg 1.1.5 of Oprea)

$\alpha(t) = (t^2, t^3)$ $\left[\text{i.e. } \alpha(t) = (t^2, t^3, 0) \in \mathbb{R}^3 \right]$

Then $\alpha'(t) = (2t, 3t^2)$

$\Rightarrow \alpha'(0) = 0$

$\therefore$ Cusp α is <u>not</u> regular (at $t=0$)

<u>Easy fact</u> (Prop. 1.1.7 of Oprea)

The curve α is a (constant speed) straight line

$\Leftrightarrow \alpha'' = 0.$

(Pf: Ex.)

Def: • The arc-length of a regular parametrized curve $\alpha: I \to \mathbb{R}^3$ from $t_0 = a$ to $t \in I$ is defined by

$$\int_a^t |\alpha'(t)|\, dt .$$

• If $I = [a,b]$, then

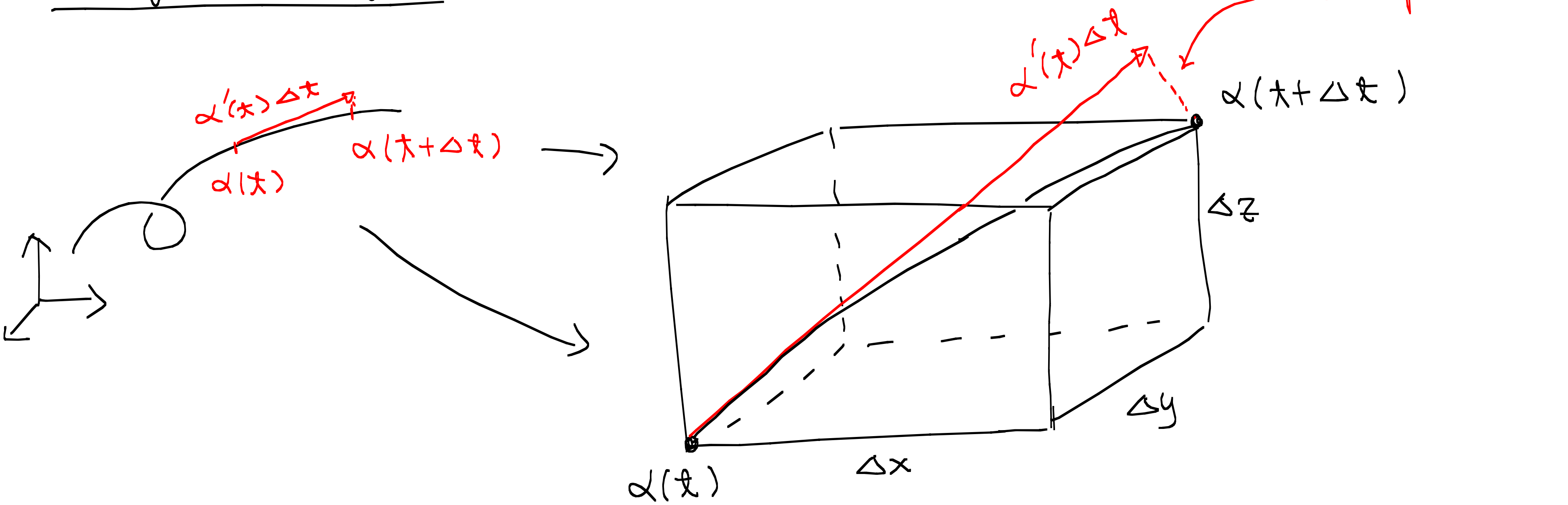
$$L(\alpha) = \int_a^b |\alpha'(t)|\, dt$$

is called the arc length of α.

(provided α is differentiable up to the end points $t=a$, $t=b$.

i.e. α is differentiable curve defined on an open interval containing the closed interval $[a,b]$.)

Idea for the definition :



$$\Rightarrow \quad |\Delta\alpha| \sim \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

$$\sim \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,\Delta t = |\alpha'(t)|\,\Delta t$$

eg 1.1.5 The arc length of $\alpha(t) = p + tv$, $v \neq 0$, from t_0 to t

is
$$s(t) = \int_{t_0}^{t} |\alpha'(t)|\,dt = \int_{t_0}^{t} |v|\,dt$$
$$= |v|(t - t_0)$$

eg 1.1.6 Circle of radius r centered at $(0,0)$:

$$\alpha(t) = (r\cos t,\ r\sin t), \quad 0 \le t \le 2\pi,$$

is a regular parametrization of the circle

$\Rightarrow$ arc length of a circle of radius r is
$$L(\alpha) = \int_0^{2\pi} |\alpha'(t)|\,dt = 2\pi r.$$

<u>Thm</u>: (Thm 1.1.11 of Oprea)

In $\mathbb{R}^3$, a line is the curve of least arc length between 2 points.

<u>Pf</u>: Let $p, q \in \mathbb{R}^3$, $p \neq q$.

Parametrizes the line between p, q by

$$\alpha(t) = p + t(q-p), \quad t \in [0,1].$$

Then $q \neq p \Rightarrow \alpha$ is regular and

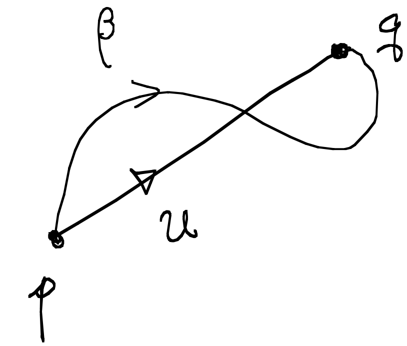
arc length $L(\alpha) = \int_0^1 |\alpha'(t)|\, dt = |q-p|$.

Suppose $\beta : [a,b] \to \mathbb{R}^3$ is another (regular) curve such that

$\beta(a) = p$ and $\beta(b) = q$.

Let $u = \dfrac{q-p}{|q-p|}$ = unit vector in the direction from p to q.

Then $\beta'(t)$ can be decomposed as



$$\beta'(t) = \langle \beta'(t), u \rangle u + v(t)$$

where $v(t)$ perpendicular to u i.e. $\langle v(t), u \rangle = 0$

($\langle a, b \rangle = a \cdot b$ the dot product of vectors a, b in $\mathbb{R}^3$.)

$$\Rightarrow \quad |\beta'(t)|^2 = \langle \beta'(t), u \rangle^2 + |v|^2$$

$$\geq \langle \beta'(t), u \rangle^2$$

$$\Rightarrow \quad |\beta'(t)| \geq |\langle \beta'(t), u \rangle| \geq \langle \beta'(t), u \rangle$$

$$\Rightarrow \quad \int_a^b |\beta'(t)| dt \geq \int_a^b \langle \beta'(t), u \rangle dt$$

$$= \left\langle \int_a^b \beta'(t)\,dt,\ u \right\rangle$$

$$= \langle \beta(b)-\beta(a),\ u \rangle$$

$$= \left\langle q-p,\ \frac{q-p}{|q-p|} \right\rangle$$

$$= |q-p|$$

i.e. $L(\beta) \geq L(\alpha)$ $\#$

Ex: Read all other examples in § 1.1 of Oprea.

1.2 Arc length parametrization

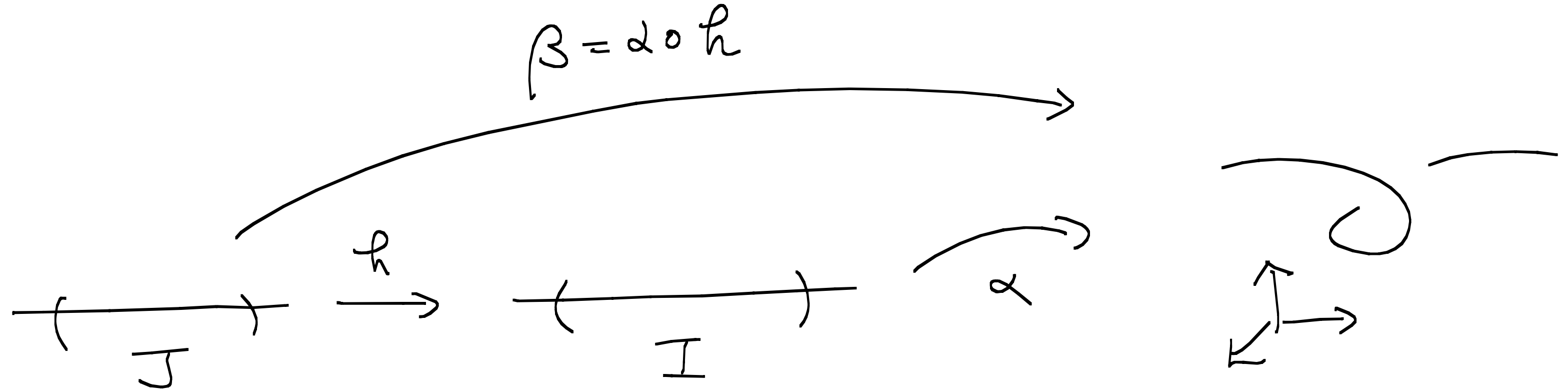
Def: Reparametrization

Let $\alpha : I \to \mathbb{R}^3$ be a curve,

$h : J \to I$ be a map of interval J one-to-one and onto I,

then $\beta = \alpha \circ h : J \to \mathbb{R}^3$

is called a reparametrization of α.



<u>Lemma</u> $\forall\, r \in J$, $\quad \beta'(r) = \alpha'(h(r))\, \frac{dh}{dr}(r)$

(Pf: Chain rule)

$\therefore\ \beta'(r)$ & $\alpha'(h(r))$ are parallel

<u>Remark</u>: Arc-length is invariant under change of parameters

i.e. $\quad L(\beta) = L(\alpha) \quad \forall\ \beta =$ reparametrization of α.

(Pf: Exercise)

<u>Thm</u> (Thm 1.2.3 of Oprea)

If α is a <u>regular</u> curve, then α may be reparametrized to have <u>unit speed</u> (or equivalently, by <u>arc-length</u>)

$$\left[\ |\alpha'| = 1 \iff \text{arc length from } \alpha(t_0) \text{ to } \alpha(t) \text{ is } \underset{\forall t}{t - t_0}\ \right]$$

Pf: Let $\alpha : [a, b] \to \mathbb{R}^3$ be a regular curve.

Define $$s(t) = \int_a^t |\alpha'(t)|\,dt \quad \Big(= \text{arc length from } \alpha(a) \text{ to } \alpha(t) \Big)$$

Then $s'(t) = |\alpha'(t)| > 0$ (since α is regular.)

$\Rightarrow$ $s(t)$ is invertible

and let $t = t(s)$ be the inverse function (defined on certain interval.)

Then the reparametrization $\beta(s) = \alpha(t(s))$

has velocity vector

$$\beta'(s) = \alpha'(t(s))\,\frac{dt}{ds}(s)$$

$$= \alpha'(t(s)) \cdot \frac{1}{\frac{ds}{dt}(t(s))}$$

$$= \frac{\alpha'(t(s))}{|\alpha'(t(s))|}$$

$$\Rightarrow \quad |\beta'(s)| = 1, \quad \forall s. \quad \#$$

<u>eg</u> (eg 1.2.4 of Oprea)

Let $\alpha(t) = (a\cos t, a\sin t, bt)$ such that $c = \sqrt{a^2+b^2} \neq 0$.

Then it is easy to get $|\alpha'(t)| = c > 0$ (constant speed)

$$\Rightarrow \quad s = s(t) = \int_0^t |\alpha'(t)|\,dt = \int_0^t c\,dt = ct$$

$$\Rightarrow \qquad t = t(s) = \frac{1}{c}\, s$$

Hence the required reparametrization with unit speed is

$$\beta(s) = \alpha(t(s)) = \alpha\left(\frac{s}{c}\right)$$

$$= \left(a \cos \frac{s}{c},\ a \sin \frac{s}{c},\ \frac{b}{c} s\right)$$

$$= \left(a \cos \frac{s}{\sqrt{a^2+b^2}},\ a \sin \frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}}\right).$$

1.3 Frenet Formulas

In this section, all curves are regular, at least 3 times differentiable and parametrized by arc-length.

Then curve $\beta : I \to \mathbb{R}^3$ (in this section) always has unit speed, i.e. tangent vector of unit length:

$$|\beta'| = 1$$

($\beta'(s) = \frac{d\beta}{ds}(s)$, & $s =$ arc-length parameter of β)

Trick to Remember: (Prop 1.3.1 of the text)

If $T(s)$ = vector field with unit length (constant), then T' is perpendicular (orthogonal) to T. i.e. $\langle T', T \rangle = 0$.

Pf: $\langle T, T \rangle = 1 \Rightarrow \langle T', T \rangle + \langle T, T' \rangle = 0$

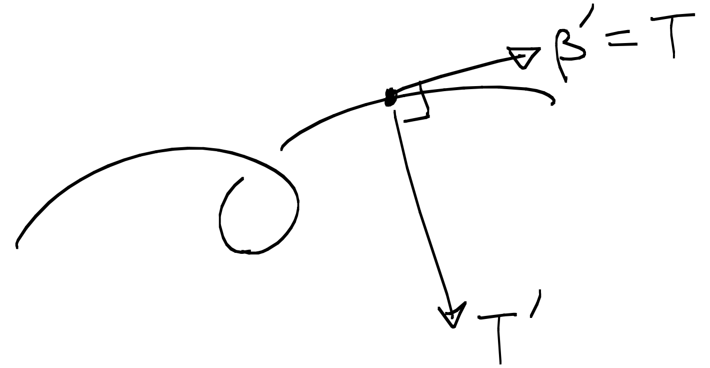
$\Rightarrow \langle T', T \rangle = 0$ #

As a consequence, we have

Prop If $\beta : I \to \mathbb{R}^3$ is parametrized by arc-length and $T = \beta'$.

Then $T'(s)$ is perpendicular to $T(s)$, $\forall s \in I$.

(We say that T' is normal to β.)



$(|\beta'| = |T| = 1)$ $\forall t$

(Caution: Not true for other parametrization.)

Def: Let $\beta : I \to \mathbb{R}^3$ be a twice diff. regular curve parametrized by arc-length s.

We define the curvature of β at s to be

$$\boxed{\kappa(s) = |T'(s)|}$$

where $T(s) = \beta'(s)$. (i.e. $\kappa(s) = |\beta''(s)|$.)

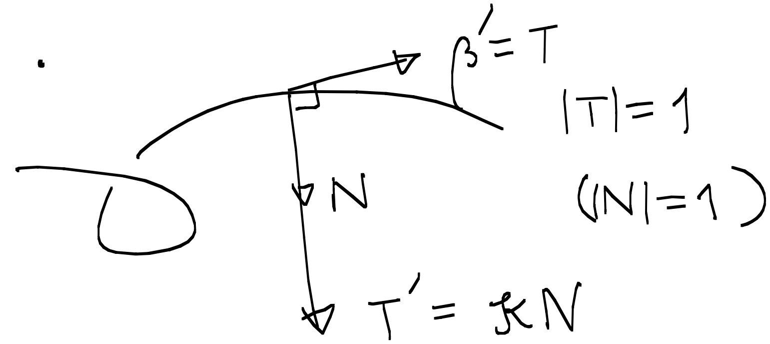
Remark: $\kappa(s) \geq 0$ for space curves as we defined.

However, we can define signed curvature for plane curves later.

Def: Assumptions and notations as in the previous definition.
We define the principal normal vector along β to be

$$\boxed{N = \frac{1}{\kappa} T'} \quad \left(= \frac{T'}{|T'|} = \frac{\beta''}{|\beta''|} \right)$$

provided $\kappa(s) \neq 0$ for $s \in I$.



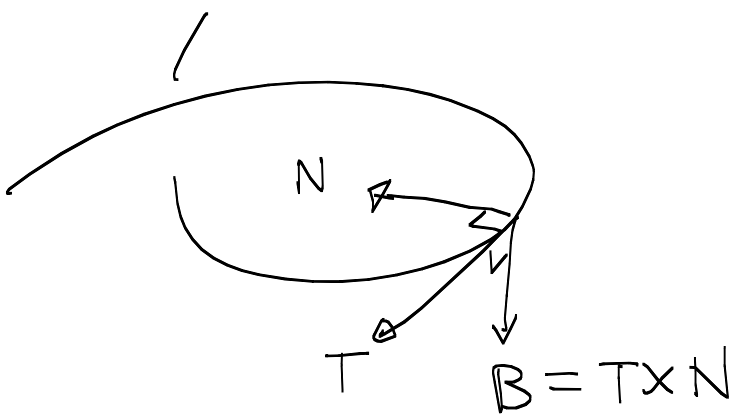
Def: Assumptions & notations as in the previous definition.

We define the binormal along β to be

$$\boxed{B = T \times N}$$

(the cross product of T & N in $\mathbb{R}^3$)

($B = T \wedge N$ in do Carmo)



By properties of cross product,

$$|B| = 1, \quad \langle B, T \rangle = \langle B, N \rangle = 0$$

$\therefore$ T, N, B are mutually perpendicular unit vectors in $\mathbb{R}^3$, i.e. $\{T, N, B\}$ is an <u>orthonormal</u> <u>basis</u> of $\mathbb{R}^3$, $\forall s \in I$.

<u>Def</u>: $\{T, N, B\}$ is called the <u>Frenet frame</u> along β.

<u>Thm</u> : (Thm 1.3.9 of Oprea)

Let $\beta : I \to \mathbb{R}^3$ be a 3-times differentiable regular curve parametrized by arc-length with nonzero curvature $\kappa > 0$.

Then the Frenet frame $\{T, N, B\}$ along β satisfies

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N \end{cases}.$$

where $\tau = -\langle B', N \rangle$ is called the <u>torsion</u> of β.

Pf: The 1^{st} eqt $T' = \kappa N$ follows directly from the definition of the normal N.

As $\{T, N, B\}$ forms an orthonormal basis for $\mathbb{R}^3$, we have

$$\begin{cases} N' = \langle N', T\rangle T + \langle N', N\rangle N + \langle N', B\rangle B \\ B' = \langle B', T\rangle T + \langle B', N\rangle N + \langle B', B\rangle B \end{cases}$$

Since $|N| = |B| = 1$, we have $\langle N', N\rangle = 0 = \langle B', B\rangle$.

$$\therefore \begin{cases} N' = \langle N', T\rangle T \qquad\qquad + \langle N', B\rangle B \\ B' = \langle B', T\rangle T - \tau N \end{cases}$$

(using $\tau = -\langle B', N\rangle$)

Now $\langle T, B \rangle = 0 \Rightarrow \langle T', B \rangle + \langle T, B' \rangle = 0$

$$\Rightarrow \langle B', T \rangle = -\langle \kappa N, B \rangle$$

$$= -\kappa \langle N, B \rangle = 0$$

$\therefore$ $B' = -\tau N$ (this is the last formula.)

Similarly, $\langle N, T \rangle = 0 \Rightarrow \langle N', T \rangle + \langle N, T' \rangle = 0$

$$\Rightarrow \langle N', T \rangle = -\langle N, \kappa N \rangle$$

$$= -\kappa .$$

$$\langle N, B \rangle = 0 \Rightarrow \langle N', B \rangle + \langle N, B' \rangle = 0$$

$$\Rightarrow \langle N', B \rangle = -\langle B', N \rangle = \tau$$

$\therefore$ $N' = -\kappa T + \tau B$. #

<u>Caution</u> : There are 2 choices of the torsion τ,

namely, $\tau = \pm \langle B', N \rangle$.

In "Oprea", $\tau = - \langle B', N \rangle$. But in "do Carmo", $\tau = \langle B', N \rangle$.

We follow Oprea's convention.

eg (eg 1.3.18 of Oprea: Helix)

$$\beta(s) = \left(a \cos \frac{s}{c},\ a \sin \frac{s}{c},\ \frac{bs}{c} \right), \text{ where } c = \sqrt{a^2+b^2},\ \underline{\underline{a>0}}$$

Then $T = \beta' = \left(-\frac{a}{c} \sin \frac{s}{c},\ \frac{a}{c} \cos \frac{s}{c},\ \frac{b}{c} \right)$ $\left(\Rightarrow |T| = 1 \right)$

$$T' = \beta'' = \left(-\frac{a}{c^2} \cos \frac{s}{c},\ -\frac{a}{c^2} \sin \frac{s}{c},\ 0 \right)$$

$$\Rightarrow \quad \kappa = |T'| = \frac{a}{c^2} > 0$$ ($\Rightarrow$ one can define principal normal)

$$\Rightarrow \quad N = \frac{T'}{\kappa} = \left(-\cos\frac{s}{c},\ -\sin\frac{s}{c},\ 0\right) \quad \text{and}$$

$$B = T \times N = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c} \\ -\cos\frac{s}{c} & -\sin\frac{s}{c} & 0 \end{vmatrix}$$

$$= \left(\frac{b}{c}\sin\frac{s}{c},\ -\frac{b}{c}\cos\frac{s}{c},\ \frac{a}{c}\right)$$

$$\Rightarrow \quad B' = \left(\frac{b}{c^2}\cos\frac{s}{c},\ \frac{b}{c^2}\sin\frac{s}{c},\ 0\right)$$

$$= \frac{b}{c^2}\left(\cos\frac{s}{c},\ \sin\frac{s}{c},\ 0\right) = -\frac{b}{c^2} N$$

$\therefore \quad \tau = \dfrac{b}{c^2}$ (exercise 1.3.19 of Oprea)

One can also verifies

$$N' = -\kappa T + \tau B. \quad (\text{Ex!})$$

If $b=0$, then $\tau=0$ and the Helix becomes a circle of radius a. And in this case,

$$\kappa = \frac{a}{a^2+b^2} = \frac{1}{a}.$$

$$\therefore \quad \kappa = \frac{1}{\text{radius}} \quad \text{for a circle.}$$

In view of our assumption on the curves in this section, we will use the term "unit speed curve" for a 3-times differentiable regular curve parametrized by arc-length in the following discussions.

Geometric meanings of curvature and torsion:

Thm (Thm 1.3.20 of Oprea)

Let $\beta(s)$ be a unit speed curve.

Then (1) $\kappa = 0 \iff \beta$ is a straight line;

(2) for $\kappa > 0$, $\tau = 0 \iff \beta$ is a plane curve.

(do Carmo, pg 24, ex 10 : "$\kappa > 0$" is needed.)

ie.

$$\left.\begin{array}{l} \text{curvature } \kappa \\ \\ \\ \text{torsion } \tau \end{array}\right\} \text{measures the deviation of a curve from being} \left\{\begin{array}{l} \text{a line;} \\ \\ \\ \text{a plane curve.} \end{array}\right.$$

Pf: (1) $\kappa = 0 \iff T' = 0$

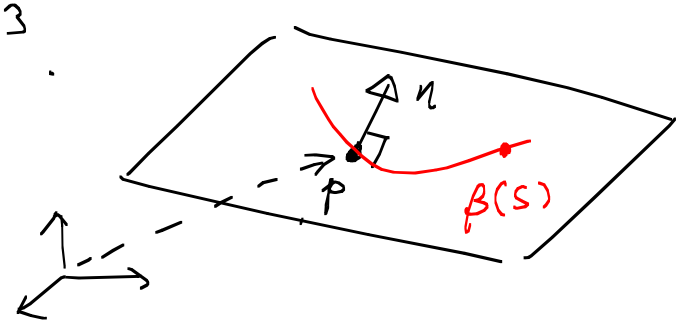
$\iff T =$ constant vector $v \in \mathbb{R}^3$ $\quad (T = \beta')$

with $|v| = |T| = 1$

$\iff \beta(s) = p + sv \quad$ for some $p \in \mathbb{R}^3$

$\iff \beta$ is a line.

(2) ($\Leftarrow$) If β is a plane curve. Let the plane containing β is normal to $\overbrace{n}^{\text{some}} \in \mathbb{R}^3$, $|n| = 1$, and passing through $\underbrace{p}_{\text{some}} \in \mathbb{R}^3$.



Then $\langle \beta(s) - p, n \rangle = 0 \quad \forall s$

Differentiation $\Rightarrow \quad \langle \beta'(s), n \rangle = 0$

(again) $\Rightarrow \quad \langle \beta''(s), n \rangle = 0$

$\therefore \quad \langle T, n \rangle = 0 = \langle \kappa N, n \rangle$

$\Rightarrow \quad n = \pm B$

which implies B is a constant vector.

$\therefore \quad B' = 0$

$\Rightarrow \quad \tau = -\langle B', N \rangle = 0.$

($\Leftarrow$) Conversely, If $\tau = 0$, the Frenet formula

$\Rightarrow B' = -\tau N = 0$

$\Rightarrow$ B is a constant vector.

Consider $f(s) = \langle \beta(s) - \beta(0), B \rangle$

Differentiation $\Rightarrow$ $f'(s) = \langle \beta'(s), B \rangle$

$$= \langle T, B \rangle = 0$$

$\therefore$ $f(s) = \text{constant} = f(0) = 0$

$\Rightarrow$ $\beta(s) \in$ plane normal to B & passing through $\beta(0)$. #

Thm (Thm 1.3.21 of Oprea)

A curve $\beta(s)$ is part of a circle $\Leftrightarrow \begin{cases} \kappa > 0 \text{ is constant } \& \\ \tau = 0 \end{cases}$

Pf : ($\Rightarrow$) Suppose β is part of a circle.

Then β is a plane curve.

$\therefore \tau = 0$

Let p_0 be the center of the circle.

Then $|\beta(s) - p_0| = r$ positive const.

$\Rightarrow \langle \beta'(s), \beta(s) - p_0 \rangle = 0 \quad \forall s.$

i.e. $\langle T, \beta(s) - p_0 \rangle = 0 \quad \forall s. \quad \text{——} (1)$

Diff. $\Rightarrow \langle T', \beta(s) - p_0 \rangle + \langle T, T \rangle = 0$

$$\therefore \quad \kappa \langle N, \beta(s) - p_0 \rangle + 1 = 0$$

$$\Rightarrow \quad \kappa > 0 \quad \text{and} \quad \langle N, \beta(s) - p_0 \rangle \neq 0 \quad \forall s$$

(κ is alway ≥ 0)

Diff. again $\Rightarrow$

$$0 = \kappa' \langle N, \beta(s) - p_0 \rangle + \kappa \left(\langle N', \beta(s) - p_0 \rangle + \cancelto{0}{\langle N, T \rangle} \right)$$

$$= \kappa' \langle N, \beta(s) - p_0 \rangle - \kappa^2 \cancelto{0}{\langle T, \beta(s) - p_0 \rangle} \quad \text{by Frenet \& (1)}$$

$$= \kappa' \langle N, \beta(s) - p_0 \rangle$$

Since $\langle N, \beta(s) - p_0 \rangle \neq 0$, we have $\kappa' = 0 \ \forall s$

$\therefore \quad \kappa > 0$ is a constant.

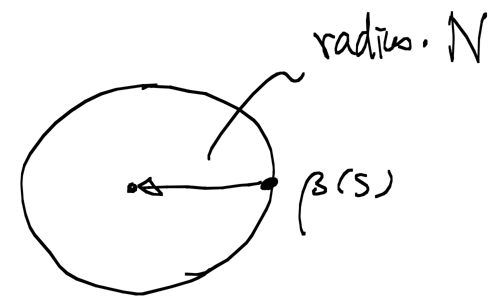
($\Leftarrow$) Conversely, if $\tau = 0$ and $\kappa > 0$. constant

We need to find a point p_0 such that

$$|\beta(s) - p_0| = r \quad \text{for some constant } r.$$

If $\beta(s)$ is a circle, then $\kappa = \frac{1}{\text{radius}}$

and $$p_0 = \beta(s) + \frac{1}{\kappa} N(s)$$



Therefore, to show that $\beta(s)$ is part of a circle, we need to show that

$$\gamma(s) = \beta(s) + \frac{1}{\kappa} N(s) \quad \text{is a constant vector.}$$

To see this, we differentiate to get

$$\gamma'(s) = \beta'(s) + \frac{1}{\kappa} N'(s) \quad \text{since } \kappa > 0 \text{ is a constant}$$

$$= T + \frac{1}{\kappa}(-\kappa T + \cancel{\tau}^{\,0} B) \quad \text{By Frenet formula}$$

$$= T - T = 0$$

$\therefore \quad \gamma(s) = p_0$ a constant vector in $\mathbb{R}^3$

And $\quad \beta(s) - p_0 = \frac{-1}{\kappa} N$

$$\Rightarrow \quad |\beta(s) - p_0| = \frac{1}{\kappa}$$

$\therefore$ $\beta(s)$ is part of the circle centered as p_0 with radius $\frac{1}{\kappa}$.

#

eg (eg 1.3.29 of Oprea)

Let $\alpha(s)$ = unit speed "closed" curve with arc length L_0

i.e. $\alpha : [0, L_0] \to \mathbb{R}^3$ is parametrized by arc-length

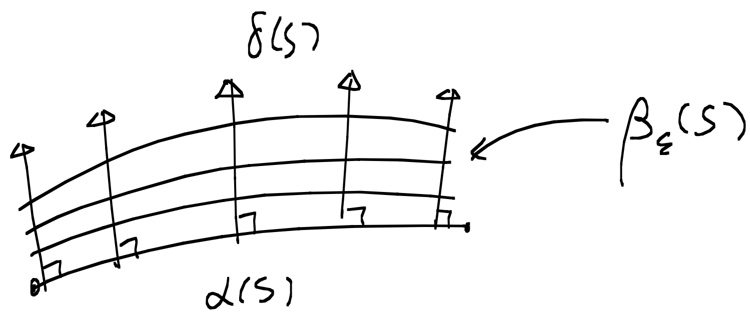
such that $\alpha(0) = \alpha(L_0)$, ...

Consider the family of curves

$$\beta_\varepsilon(s) = \alpha(s) + \varepsilon\,\delta(s)$$

In general,
($s \neq$ arc length of β_ε
for $\varepsilon > 0$.)

where $\langle \delta(s), \alpha'(s) \rangle = 0 \quad \forall s$.



Then $\quad L(\varepsilon) \overset{def}{=} L(\beta_\varepsilon) = \int |\beta'| \, ds$

$$= \int \langle \beta', \beta' \rangle^{1/2} \, ds$$

$$= \int \langle \alpha' + \varepsilon \delta', \alpha' + \varepsilon \delta' \rangle^{1/2} \, ds$$

$$= \int \left(|\alpha'|^2 + 2\varepsilon \langle \alpha', \delta' \rangle + \varepsilon^2 |\delta'|^2 \right)^{1/2} ds$$

$$\Rightarrow \quad \frac{dL}{d\varepsilon}\bigg|_{\varepsilon=0} = \int \frac{2\langle \alpha', \delta' \rangle + 2\varepsilon |\delta'|^2}{2\sqrt{|\alpha'|^2 + 2\varepsilon \langle \alpha', \delta' \rangle + \varepsilon^2 |\delta'|^2}}\bigg|_{\varepsilon=0} ds$$

$$= \int \langle \alpha', \delta' \rangle \, ds \qquad (\text{since } |\alpha'| = 1)$$

$$= \int \left(\cancelto{0}{\langle \alpha', \delta \rangle'} - \langle \alpha'', \delta \rangle \right) ds$$

$$= -\int \langle T', \delta \rangle \, ds$$

$$\therefore \quad \left.\frac{dL}{d\varepsilon}\right|_{\varepsilon=0} = -\int \kappa \langle N, \delta \rangle \, ds \qquad \text{by Frenet formula}$$

Note that if $\delta = N$, then $\left.\frac{dL}{d\varepsilon}\right|_{\varepsilon=0} = -\int \kappa \, ds$.

This gives another interpretation of curvature κ.

1.4 Non-unit Speed Curves

Def: Let α = curve with speed $v = |\alpha'(t)|$ (>0)

Suppose $\bar{\alpha}(s)$ is a parametrization of α by arc length $s = s(t)$.

We define

(1) The unit tangent of $\alpha(t)$ = $T(t) \overset{\text{def}}{=\!=} \bar{T}(s(t))$

(2) The curvature $\kappa(t) \overset{\text{def}}{=\!=} \bar{\kappa}(s(t))$.

(3) If $\kappa > 0$, principal normal $N(t) \overset{\text{def}}{=\!=} \bar{N}(s(t))$.

(4) If $\kappa > 0$, binormal $B(t) \overset{\text{def}}{=\!=} \bar{B}(s(t))$.

(5) If $\kappa > 0$, torsion $\tau(t) \overset{\text{def}}{=\!=} \bar{\tau}(s(t))$.

Remark : From $\alpha(t) = \bar{\alpha}(s(t))$,

$$\alpha'(t) = \frac{d}{dt}\left(\bar{\alpha}(s(t))\right)$$

$$= \frac{d\bar{\alpha}}{ds}(s(t))\,\frac{ds}{dt}(t)$$

$$= \bar{T}(s(t))\,|\alpha'(t)|$$

$\therefore\quad T(t) = \dfrac{\alpha'(t)}{|\alpha'(t)|}$ = unit vector in the direction of $\alpha'(t)$.

(The definition makes sense !)

& $\alpha'(t) = \nu T(t)$

Thm (Thm 1.4.2 of Oprea)

For a regular curve $\alpha(t)$ with speed $\nu = \dfrac{ds}{dt} = |\alpha'(t)|$ and curvature $\kappa > 0$,

$$\begin{cases} T'(t) = & \kappa\nu N \\ N'(t) = -\kappa\nu T & \qquad + \tau\nu B \\ B'(t) = & -\tau\nu N \end{cases}.$$

Pf: $\quad T(t) = \bar{T}(s(t))$

$$\Rightarrow \quad T'(t) = \frac{d\bar{T}}{ds}(s(t))\,\frac{ds}{dt}(t)$$

$$= \bar{\kappa}(s(t))\,\bar{N}(s(t))\,\nu \qquad \text{by Frenet formula}$$

$$= \kappa \nu N.$$

Similarly for $N'(t)$ & $B'(t)$ (Exercise!) $\#$

Lemma (Lemma 1.4.3 of Oprea)

$\boxed{\alpha' = \nu T}$ and $\boxed{\alpha'' = \frac{d\nu}{dt} T + \kappa \nu^2 N}$

Pf: $\alpha' = \nu T$ is proved in the remark.

The 2^{nd} is an easy exercise. $\#$

<u>Thm</u> (Thm 1.4.5 of Oprea)

For any regular curve α, the following formulas hold.

(1) $$B = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|}$$

(2) $$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}$$

(3) $$\tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|^2}$$

<u>Pf</u> : (1) & (2) By lemma $\alpha' \times \alpha'' = (\nu T) \times \left(\frac{d\nu}{dt} T + \kappa \nu^2 N \right)$

$= \kappa \nu^3 \, T \times N = \kappa \nu^3 B$

Since $|B| = 1$, we have $|\alpha' \times \alpha''| = \kappa \nu^3$

$$\therefore \quad B = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|} \quad \text{and} \quad \kappa = \frac{|\alpha' \times \alpha''|}{\nu^3} = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}.$$

(3) $\alpha'' = \nu' T + \kappa \nu^2 N$

$$\Rightarrow \alpha''' = \nu'' T + \nu' T' + (\kappa \nu^2)' N + \kappa \nu^2 N'$$

$$= \nu'' T + \nu'(\kappa \nu N) + (\kappa \nu^2)' N + \kappa \nu^2 (-\kappa \nu T + \tau \nu B)$$

$$= (\nu'' - \kappa^2 \nu^3) T + [\nu' \kappa \nu + (\kappa \nu^2)'] N + \kappa \tau \nu^3 B$$

$$\Rightarrow \kappa \tau \nu^3 = \langle B, \alpha''' \rangle$$

$$\Rightarrow |\alpha' \times \alpha''| \tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|}$$

$$\Rightarrow \quad \tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|^2}$$

eg (Involute and Plane Evolutes, exercise 1.2.7 & eg. 1.4.11 of Oprea)

Let $\alpha(t)$ be a regular plane curve with an arc-length function $s(t)$.

We define:

- (an) Involute of α: $\mathcal{I}(t) = \alpha(t) - s(t)\, T(t)$ ← unit tangent of α

  (depends on the choice of $s(t)$)

- (Plane) Evolute of α: $\mathcal{E}(t) = \alpha(t) + \frac{1}{\kappa(t)} N(t)$ ← principal normal of α.

Since $\mathcal{E}_\alpha(t)$ is also a curve, one can consider the involute $\mathcal{J}_{\mathcal{E}_\alpha}(t)$ of $\mathcal{E}_\alpha(t)$.

We <u>claim</u> that for an arc-length function $s_{\mathcal{E}_\alpha}(t)$,

$$\mathcal{J}_{\mathcal{E}_\alpha}(t) = \alpha(t) \qquad \text{(provided } \mathcal{E}_\alpha \text{ is regular)}$$

That is an involute of evolute of α is α.

"<u>Pf</u>": To calculate the involute of $\mathcal{E}(t)$, we need to find unit tangent vector $T_{\mathcal{E}}(t)$ of $\mathcal{E}(t)$ and an arc-length function $s_{\mathcal{E}}(t)$ of $\mathcal{E}(t)$.

By definition, $\mathcal{E} = \alpha + \frac{1}{\kappa} N$

$$\Rightarrow \quad \mathcal{E}' = \alpha' + \left(\frac{1}{\kappa}\right)' N + \frac{1}{\kappa} N'$$

$$= \alpha' + \left(\frac{1}{\kappa}\right)' N + \frac{1}{\kappa}(-\kappa T + \tau B)$$

Since α is a plane curve, $\tau = 0$.

(provided $\kappa > 0$)

$$\Rightarrow \quad \varepsilon' = \alpha' + \left(\frac{1}{\kappa}\right)' N - \nu T$$

$$= \left(\frac{1}{\kappa}\right)' N$$

$$\therefore \quad T_{\varepsilon}(t) = \frac{\varepsilon'}{|\varepsilon'|} = N(t)$$ the principal normal of α,

(assume $\left(\frac{1}{\kappa}\right)' > 0$)

If ε is regular, $|\varepsilon'| = \left|\left(\frac{1}{\kappa}\right)'\right| \neq 0$.

$$\therefore \quad \int_0^t |\varepsilon'|\, dt = \int_0^t \left|\left(\frac{1}{\kappa}\right)'\right| dt$$

$$= \int_0^t \left(\frac{1}{\kappa}\right)' dt = \left(\frac{1}{\kappa} - \frac{1}{\kappa(0)}\right)$$

Hence $\frac{1}{\kappa(t)}$ is an arc-length function and can be chosen as $S_{\varepsilon}(t)$.

Then the involute of $\mathcal{E}$ with the arc-length function $S_{\mathcal{E}} = \frac{1}{\kappa}$

is $\quad \mathcal{J}_{\mathcal{E}}(t) = \mathcal{E}(t) - S_{\mathcal{E}}(t)\, T_{\mathcal{E}}(t)$

$$= \left(\alpha + \frac{1}{\kappa} N\right) - \left(\frac{1}{\kappa}\right) N$$

$$= \alpha(t). \quad \#$$

eg (eg 1.4.12 of Oprea)

Let $\alpha(t) = \left(\frac{t^2}{2a}, t\right)$ (a parabola in xy-plane)
$(a > 0)$

Then $\alpha'(t) = \left(\frac{t}{a}, 1\right)$

$$\nu = |\alpha'(t)| = \sqrt{\frac{t^2}{a^2} + 1} = \frac{\sqrt{a^2 + t^2}}{a}$$

$$\Rightarrow \quad T = \left(\frac{t}{\sqrt{a^2+t^2}}, \frac{a}{\sqrt{a^2+t^2}} \right).$$

Hence $$T' = \left(\frac{a^2}{(a^2+t^2)^{3/2}}, -\frac{at}{(a^2+t^2)^{3/2}} \right)$$

By Frenet's formula $\quad T' = \kappa \nu N$

$$\Rightarrow \quad |T'| = \kappa \nu$$

i.e.
$$\frac{a}{a^2+t^2} = \kappa \frac{\sqrt{a^2+t^2}}{a}$$

$$\Rightarrow \quad \kappa = \frac{a^2}{(a^2+t^2)^{3/2}}$$

This then gives $\quad N = \frac{a^2+t^2}{a} \left(\frac{a^2}{(a^2+t^2)^{3/2}}, \frac{-at}{(a^2+t^2)^{3/2}} \right)$

$$= \left(\frac{a}{\sqrt{a^2+t^2}}, \frac{-t}{\sqrt{a^2+t^2}} \right)$$

Finally, we can calculate

$$\varepsilon(t) = \alpha(t) + \frac{1}{\kappa(t)} N(t)$$

$$= \left(\frac{t^2}{2a}, t \right) + \frac{(a^2+t^2)^{3/2}}{a^2} \left(\frac{a}{\sqrt{a^2+t^2}}, \frac{-t}{\sqrt{a^2+t^2}} \right),$$

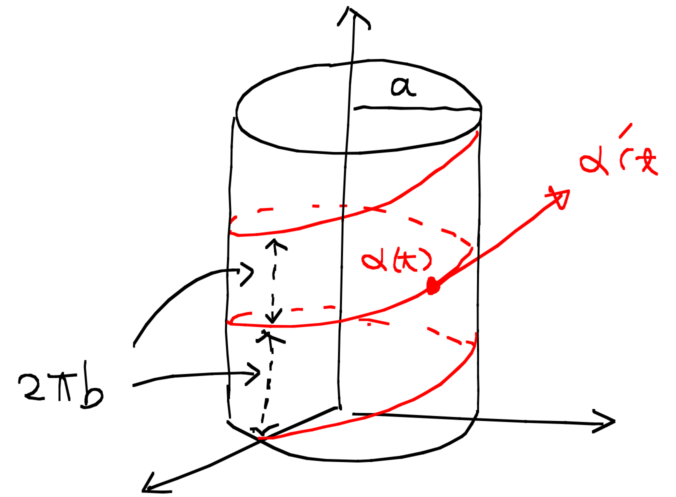
$$= \left(\frac{2a^2+3t^2}{2a}, -\frac{t^3}{a} \right). \quad \#$$

1.5 Some Implications of Curvature and Torsion

Recall that the (standard) helix is

$$\alpha(s) = \left(a\cos\frac{s}{c},\ a\sin\frac{s}{c},\ \frac{b}{c}s\right)$$

where $c = \sqrt{a^2+b^2}$.



Its unit tangent is

$$T = \left(-\frac{a}{c}\sin\frac{s}{c},\ \frac{a}{c}\cos\frac{s}{c},\ \frac{b}{c}\right)$$

The axis of rotation is $e_3 = (0,0,1)$.

$$\Rightarrow \quad \langle T, e_3\rangle = \frac{b}{c} \text{ is a constant.}$$

To generalize this, we make the following

Def: A unit speed curve α is a <u>cylindrical helix</u> if $\exists$ <u>constant</u> <u>unit</u> vector u such that

$$\langle T, u \rangle = \text{const. along the curve } \alpha.$$

Thm (Thm 1.5.2 of Oprea)

Suppose α has curvature $\kappa > 0$. Then α is a cylindrical helix $\Leftrightarrow$ τ/κ is constant.

Pf: ($\Rightarrow$) If $\exists$ constant unit vector u, ($|u| = 1$) such that $\langle T, u \rangle = \text{constant}$.

Then we may write

$$\langle T, u \rangle = \cos\theta \quad \text{for some constant } \theta$$

as $|T| = |u| = 1$.

Differentiation $\Rightarrow$ $0 = \langle T', u \rangle$

$= \langle \kappa N, u \rangle$ (by Frenet's formula)

Since $\kappa > 0$, $\langle N, u \rangle = 0$.

$\Rightarrow$ $u = \langle u, T \rangle T + \langle u, B \rangle B$.

$\Rightarrow$ $u = \cos\theta\, T + \sin\theta\, B$ (change θ to $-\theta$ if necessary)

Diff

$\Rightarrow$ $0 = \cos\theta\, T' + \sin\theta\, B'$ ($\theta =$ const.)

$= \cos\theta \cdot \kappa N + \sin\theta \cdot (-\tau N)$ (Frenet)

$= (\kappa \cos\theta - \tau \sin\theta) N$

$$\Rightarrow \qquad \frac{\tau}{\kappa} = \cot\theta = \text{const.}$$

($\Leftarrow$) Conversely, if $\tau/\kappa = \text{constant}$, $(\kappa > 0)$

Then $\exists$ constant θ such that

$$\tau/\kappa = \cot\theta$$

(just take a branch of $\cot^{-1}$)

Define $u = \cos\theta\, T + \sin\theta\, B$.

Then $|u|^2 = \cos^2\theta + \sin^2\theta = 1$ as $\{T, N, B\}$ orth. normal

and

$$\begin{aligned} u' &= \cos\theta\, T' + \sin\theta\, B' \\ &= \cos\theta \cdot \kappa N + \sin\theta\,(-\tau N) \\ &= (\kappa\cos\theta - \tau\sin\theta)\, N = 0 \end{aligned}$$

$\Rightarrow$ u is a unit constant vector.

By construction of u, we have $\langle T, u \rangle = \cos\theta$
$= \text{const.}$

$\#$

To generalize the above theorem, we define

<u>Def</u>: A unit speed curve $\alpha(s)$ is <u>rectifying</u> if $\exists$ fixed point $p_0 \in \mathbb{R}^3$ such that

$$\alpha(s) - p_0 \in \text{the } TB\text{-plane at } \alpha(s).$$

called the rectifying plane

(i.e. $\alpha(s) - p_0$ is a linear combination of T & B only, no N)

<u>Thm</u> (Thm 1.5.11 of Oprea)

Suppose α has curvature $\kappa > 0$.

Then α is rectifying $\Longleftrightarrow$ $\frac{\tau}{\kappa} = as + b$ for some constants a, b with $a \neq 0$.

<u>Pf</u>: ($\Rightarrow$) If α is rectifying,

then $\exists$ functions $\lambda(s)$, $\mu(s)$ and a fixed pt. $p_0 \in \mathbb{R}^3$

such that $$\alpha - p_0 = \lambda T + \mu B .$$

Diff $\Rightarrow$ $$\alpha' = \lambda' T + \lambda T' + \mu' B + \mu B'$$

$$\Rightarrow \quad T = \lambda' T + \lambda \kappa N + \mu' B - \mu \tau N$$

$$\Rightarrow \quad \lambda' = 1 , \quad \mu' = 0 , \quad \lambda \kappa - \mu \tau = 0$$

since $\{T, N, B\}$ is a basis.

$$\Rightarrow \quad \lambda = s + c_1 , \quad \mu = c_2$$

If $c_2 = 0$, then $\mu = 0$

$$\Rightarrow \quad \lambda \kappa = 0 \Rightarrow \lambda = 0 \qquad \text{as } \kappa > 0 .$$

This contradicts $\lambda = s + c_1$. $\therefore$ $c_2 \neq 0$

Hence $$\frac{\tau}{\kappa} = \frac{\lambda}{\mu} = \left(\frac{1}{c_2}\right) s + \left(\frac{c_1}{c_2}\right) .$$

($\Leftarrow$) Conversely, if $\frac{\tau}{\kappa} = as+b$ for some consts. $a \neq 0$ & b.

Rewrite $\frac{\tau}{\kappa} = \dfrac{s+(\frac{b}{a})}{(\frac{1}{a})}$ and then easily see that

one should check

$$\frac{d}{ds}\left(\alpha - (s+\tfrac{b}{a})T - (\tfrac{1}{a})B\right).$$

Straight forward calculation gives

$$\frac{d}{ds}\left(\alpha - (s+\tfrac{b}{a})T - (\tfrac{1}{a})B\right)$$

$$= \alpha' - T - (s+\tfrac{b}{a})T' - \tfrac{1}{a}B'$$

$$= 0 - (s+\tfrac{b}{a})\kappa N + \tfrac{1}{a}\tau N$$

$$= \frac{\kappa}{a}\left(-(as+b) + \frac{\tau}{\kappa}\right)N = 0$$

$\therefore$ $\alpha - (s+\frac{b}{a})T - \frac{1}{a}B = p_0$ a fixed point in $\mathbb{R}^3$. ※

Supplementary section : <u>Local Canonical Form</u> (do Carmo)

Let $\alpha(s) = (\alpha^1(s), \alpha^2(s), \alpha^3(s))$ be a unit speed curve.

Then Frenet's formula implies

$$\alpha' = T \qquad\qquad (\kappa(0) > 0)$$

$$\alpha'' = T' = \kappa N$$

$$\alpha''' = \kappa' N + \kappa N' = \kappa' N + \kappa(-\kappa T + \tau B)$$

$$= -\kappa^2 T + \kappa' N + \kappa\tau B$$

Suppose the coordinate system of $\mathbb{R}^3$ is chosen so that

$$\begin{cases} \alpha(0) = (0,0,0) \\ T(0) = e_1, \quad N(0) = e_2, \quad B(0) = e_3 \end{cases}$$

Then Taylor expansion $\Rightarrow$

$$\alpha(s) = \alpha(0) + \alpha'(0)s + \alpha''(0)\frac{s^2}{2} + \alpha'''(0)\frac{s^3}{6} + \text{higher order terms}$$

$$= sT(0) + \frac{s^2}{2}\kappa(0)N(0) + \frac{s^3}{6}\left(-\kappa^2(0)T(0) + \kappa'(0)N(0) + \kappa(0)\tau(0)B(0)\right) + \cdots$$

$$= \left(s - \frac{\kappa^2(0)}{6}s^3\right)e_1 + \left(\frac{\kappa(0)}{2}s^2 + \frac{\kappa'(0)}{6}s^3\right)e_2 + \frac{\kappa(0)\tau(0)}{6}s^3 e_3 + \cdots$$

i.e.

$$(*)\begin{cases} \alpha^1(s) = s - \dfrac{\kappa^2(0)}{6}s^3 + \cdots \\ \alpha^2(s) = \dfrac{\kappa(0)}{2}s^2 + \dfrac{\kappa'(0)}{6}s^3 + \cdots \\ \alpha^3(s) = \dfrac{\kappa(0)\tau(0)}{6}s^3 + \cdots \end{cases}$$

where $\cdots$ mean remainder terms satisfying

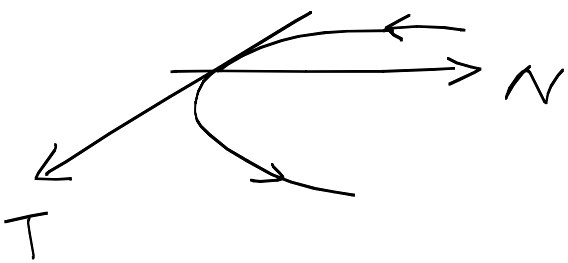
$$\lim_{s\to 0} \cdots / s^3 = 0 .$$

$(*)$ is called the local canonical form of α in a neighborhood of $s=0$

- Hence the projection of α on TN-plane (osculating plane) is locally given by

$$\begin{cases} x = \alpha^1(s) = s - \dfrac{\kappa^2(0)}{6} s^3 + \cdots \\ y = \alpha^2(s) = \dfrac{\kappa(0)}{2} s^2 + \dfrac{\kappa'(0)}{6} s^3 + \cdots \end{cases}$$

$$\Rightarrow \quad y = \frac{\kappa(0)}{2} x^2 + \cdots \quad \text{like a } \underline{\text{parabola}}$$



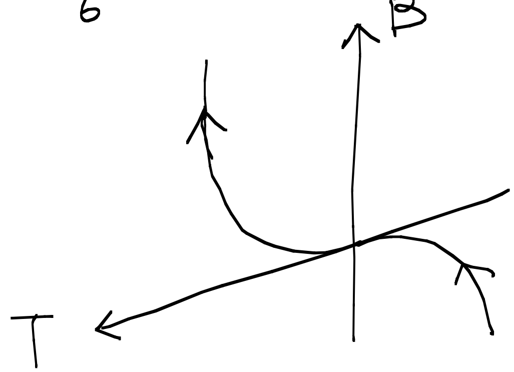
$(\kappa(0)>0)$

Projection on TN-plane (osculating plane)

- The projection on TB-plane (rectifying plane):

$$\begin{cases} x = \alpha^1(s) = s - \dfrac{\kappa^2(0)}{6} s^3 + \cdots \\ z = \alpha^3(s) = \dfrac{\kappa(0)\tau(0)}{6} s^3 + \cdots \end{cases}$$

$$\Longrightarrow \quad z = \frac{\kappa(0)\tau(0)}{6} x^3 + \cdots$$ like a cubic curve



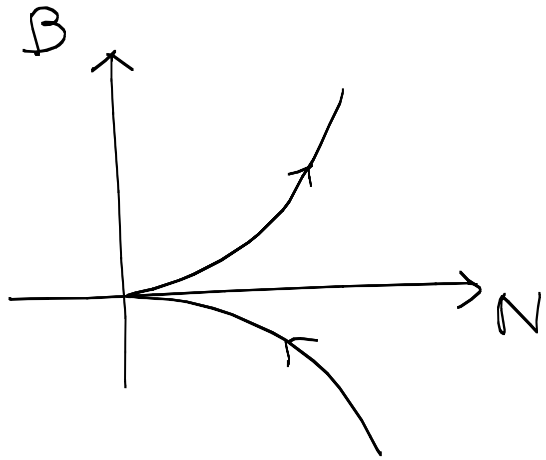
projection on TB plane (rectifying plane)

(figure shows the case $\kappa(0)>0$, $\tau(0)>0$)

- Projection on NB-plane (normal plane):

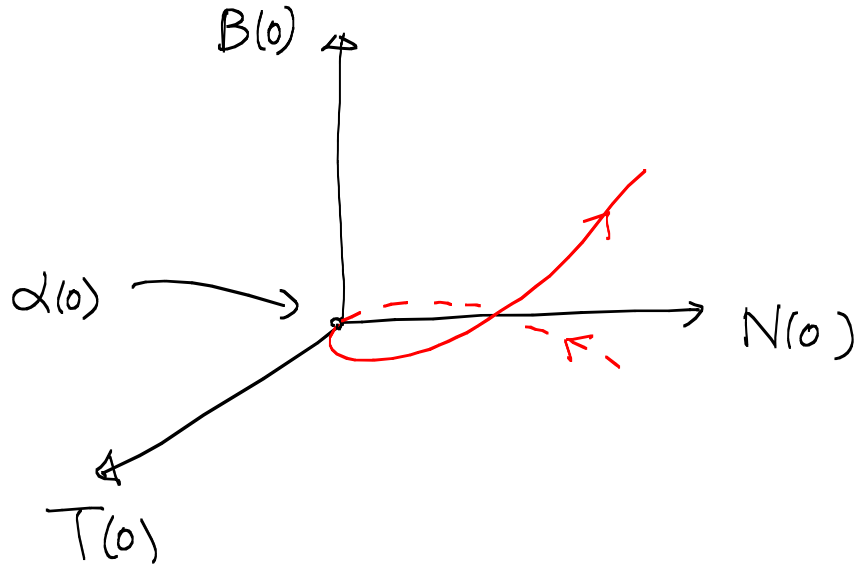
$$\begin{cases} y = \alpha^2(s) = \dfrac{\kappa(0)}{2} s^2 + \dfrac{\kappa'(0)}{6} s^3 + \cdots \\ z = \alpha^3(s) = \dfrac{\kappa(0)\tau(0)}{6} s^3 + \cdots \end{cases}$$

like a <u>cusp</u> $\sim (t^2, t^3)$. $\left(\begin{array}{l} \text{if } \kappa(0)>0,\ \tau(0)>0 \\ \text{otherwise} \sim (t^2, -t^3) \end{array} \right)$



Projection on NB-plane (normal plane)

All together, the curve α looks like



(figure shows the case of $\kappa(0)>0$, $\tau(0)>0$)

1.6 Green's Theorem and the Isoperimetric Inequality

In this section, we consider only simple closed plan curves.

Recall: • A closed (smooth) plane curve is a (parametrized) curve

$\alpha : [a,b] \to \mathbb{R}^2$ such that $\alpha^{(n)}(a) = \alpha^{(n)}(b) \quad \forall\, n \geq 0$,

where $\alpha^{(n)}$ = n-th derivative of α.

• And a curve is simple if it has no self-intersection.

A simple closed plan curve enclosing a simply connected region (ie. region without holes) and we have the following thm in Advanced Calculus:

Thm (Green's Thm, Thm 1.6.2 of Oprea)

Let $\mathcal{R}$ = simply connected region with boundary curve C in $\mathbb{R}^2$,

$P(x,y)$, $Q(x,y)$ = smooth functions on $\mathcal{R}$.

Then $$\iint_{\mathcal{R}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy = \int_C P\,dy - Q\,dx.$$

Cor (Example 1.6.4 of Oprea)

$$\text{Area}(\mathcal{R}) = \frac{1}{2}\int_C x\,dy - y\,dx = \int_C x\,dy = \int_C -y\,dx$$

(pf : Take $P = \frac{x}{2}$, $Q = \frac{y}{2}$ in the above thm to get the 1st equality.
Similarly for others.)

Using Green's thm, we are going to prove

Thm (The Isoperimetric Inequality) (Thm 1.6.1 of Oprea)

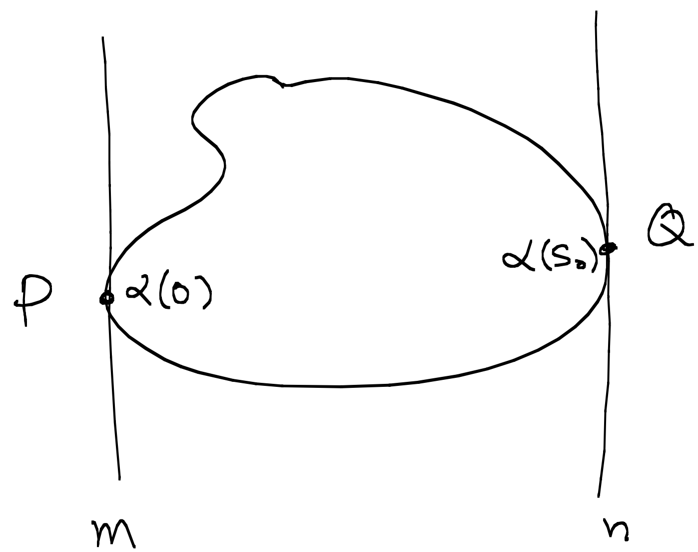
Among all simple closed curves in the plane having a fixed length, the circle bounds the enclosed region of largest area. i.e. if C = a simple closed curve with length L enclosing an area A,

then $\boxed{L^2 \geq 4\pi A.}$

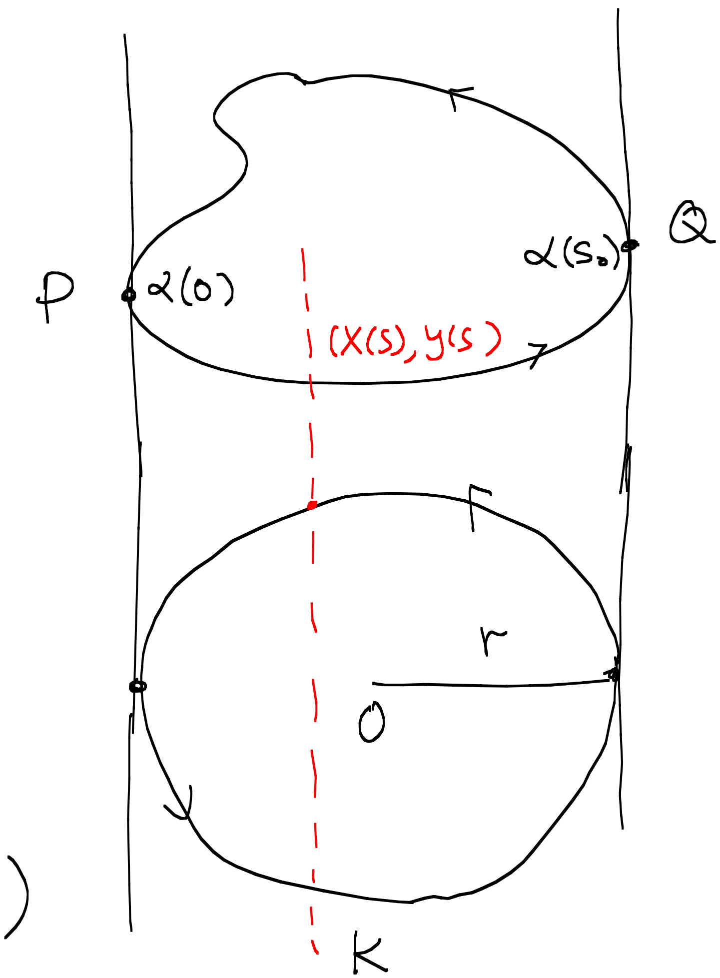
And the equality holds $\Leftrightarrow$ C = circle.

Pf: Let C be parametrized by $\alpha(s) = (x(s), y(s))$ for $0 \le s \le L$ by arc-length s.

Since C is a closed curve, we may assume C is between 2 vertical straight lines m & n such that m, n touch C at points $P = \alpha(0)$ & $Q = \alpha(s_0)$ respectively.



We constructe a comparsion circle K inside and tangent to the lines.



Let r = radius of K

$= \frac{1}{2} \times$ distance between the lines

The circle K can be parametrized by

$\beta(s) = (\beta_1(s), \beta_2(s))$ for $0 \le s \le L$ with

$\beta_1(s) = x(s)$ (the x-coordinate of $\alpha(s)$)

$$\beta_2(s) = \begin{cases} -\sqrt{r^2 - x(s)^2}, & 0 \le s \le s_0 \\ +\sqrt{r^2 - x(s)^2}, & s_0 \le s \le L \end{cases}$$

Note that $x(0) = x(L) = -r \Rightarrow \beta_2(0) = \beta_2(L) = 0$ and hence $\beta(0) = \beta(L)$.

For C, Green's thm $\Rightarrow$ area enclosed by C is

$$A_C = \int_C x\,dy = \int_0^L x(s)\,y'(s)\,ds \,. \qquad \left(\begin{array}{l}\text{2nd equality} \\ \text{in Cor.}\end{array}\right)$$

For the circle K,

$$\pi r^2 = A_K = \int_K -\beta_2(s)\,dx = \int_0^L -\beta_2(s)\,x'(s)\,ds. \qquad \left(\begin{array}{l}\text{3rd equality} \\ \text{in Cor.}\end{array}\right)$$

$$\therefore \quad A_C + \pi r^2 = \int_0^L [x(s)y'(s) - \beta_2(s)x'(s)]\,ds$$

$$= \int_0^L \langle (x, \beta_2), (y', -x') \rangle\,ds$$

$$(1) \longrightarrow \leq \int_0^L |(x, \beta_2)|\,|(y', -x')|\,ds \qquad \text{(Cauchy-Schwarz)}$$

$$= \int_0^L (x^2+\beta_2^2)^{1/2} (y'^2+x'^2)^{1/2} ds$$

$$= \int_0^L r\, ds$$

$$= rL$$

$\left(\begin{array}{l} x'^2+y'^2=1 \text{ unit speed} \\ x^2+\beta_2^2 = x^2+r^2-x^2 = r^2, \end{array} \right)$

(2) $\therefore$ $\pi r^2 - Lr + A_C \leq 0$ (quadratic ineq.)

$\Rightarrow$ $L^2 - 4\pi A_C \geq 0$. This is the Isoperimetric inequality.

For the equality case, it is clear that if C is circle of radius r, then $L^2 = (2\pi r)^2 = 4\pi(\pi r^2) = 4\pi A_C$.

Conversely, if $L^2 = 4\pi A_C$,

then all the inequalities become equalities.

Hence by (2) $\quad \pi r^2 - Lr + A_c = 0$ with $L^2 = 4\pi A_c$

$$\Rightarrow \quad r = \frac{L}{2\pi} \quad \left(\therefore\ r \text{ is indep. of the choice of the parallel lines man} \right)$$

a constant

By (1), $\quad (x, \beta_2) = \lambda (y', -x')$ for some $\lambda > 0$

(equality case of Cauchy - Schwarz)

Since $\quad r^2 = x^2 + \beta_2^2 \quad \& \quad x'^2 + y'^2 = 1$,

$\lambda = r$ is a constant

$$\Rightarrow \quad \begin{cases} x = r y' \\ \beta_2 = -r x' \end{cases} \quad \text{where } r = \frac{L}{2\pi}.$$

Now for $0 \leq s \leq s_0$,

$$-\sqrt{r^2 - x^2} = -r x'$$

$$\Rightarrow \quad \frac{x'}{\sqrt{r^2 - x^2}} = \frac{1}{r}$$

$$\Rightarrow \quad \int \frac{dx}{\sqrt{r^2 - x^2}} = \int \frac{x' ds}{\sqrt{r^2 - x^2}} = \int \frac{ds}{r}$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{r}\right) = \frac{s}{r} + d \qquad \text{for some constant } d.$$

$$\Rightarrow \quad x = r \sin\left(\frac{s}{r} + d\right)$$

Since $\begin{cases} x(0) = -r, \\ x(s_0) = r \end{cases}$, we have

$$-1 = \sin d \Rightarrow d = -\frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$$

$$\therefore \quad x = r \sin\left(\frac{s}{r} - \frac{\pi}{2}\right)$$

$$\Rightarrow \quad x = -r \cos\left(\frac{s}{r}\right)$$

Hence $\quad r = -r\cos\left(\frac{s_0}{r}\right)$

$$\Rightarrow \quad s_0 = \pi r$$

Putting $x = -r\cos\left(\frac{s}{r}\right)$ into $\quad x = r y'$

$$\Rightarrow \quad -\cos\left(\frac{s}{r}\right) = y'$$

$$\Rightarrow \quad y = -r\sin\frac{s}{r} + d_1 \qquad \text{for some constant } d_1$$

(after translation)

Using $y(0) = 0$, we have $d_1 = 0$

(Check $0 = y(s_0) = -r\sin\left(\frac{s_0}{r}\right) = -r\sin\left(\frac{\pi r}{r}\right)$ ✓)

$$\therefore \quad (x, y) = r\left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right)$$

$\Rightarrow$ For $0 \le s \le s_0 = \pi r$, C is the lower semi-circle.

Similarly, $\pi r = s_0 \le s \le L = 2\pi r$, C is the upper semi-circle.

$\therefore$ C is a circle of radius $\frac{L}{2\pi}$. ※

Ch 2 Surfaces

2.1 Definition

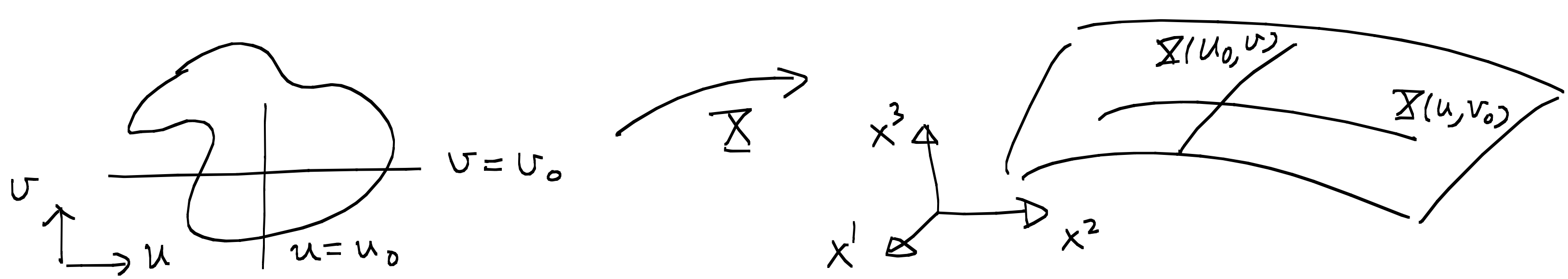
Let
- D = (connected) open set in $\mathbb{R}^2$,
- $\mathbf{X} : D \to \mathbb{R}^3$

  $(u,v) \mapsto (x^1(u,v),\ x^2(u,v),\ x^3(u,v))$

  be a mapping of D into 3-space.

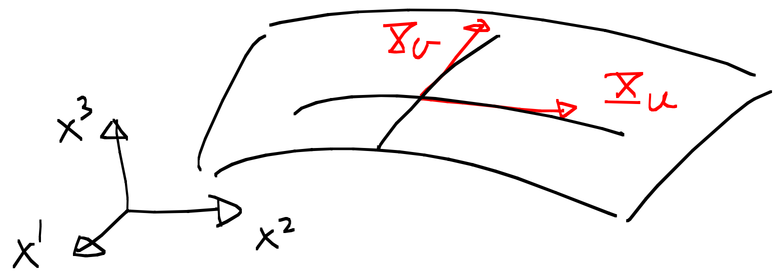
- $x^i(u,v)$ = component functions of $\mathbf{X}$
- Fix $v = v_0$, $\mathbf{X}(u, v_0)$ is called the <u>u-parameter curve</u>

  Fix $u = u_0$, $\mathbf{X}(u_0, v)$ is called the <u>v-parameter curve</u>



- Tangent vectors for u-parameter and v-parameter curves are given by

$$\begin{cases} X_u = \left(\dfrac{\partial x^1}{\partial u}, \dfrac{\partial x^2}{\partial u}, \dfrac{\partial x^3}{\partial u} \right) \\[2ex] X_v = \left(\dfrac{\partial x^1}{\partial v}, \dfrac{\partial x^2}{\partial v}, \dfrac{\partial x^3}{\partial v} \right) \end{cases}$$



__Def__: $X : D^{\subset \mathbb{R}^2} \longrightarrow \mathbb{R}^3$ is __regular__ if

$$X_u \times X_v \neq 0$$

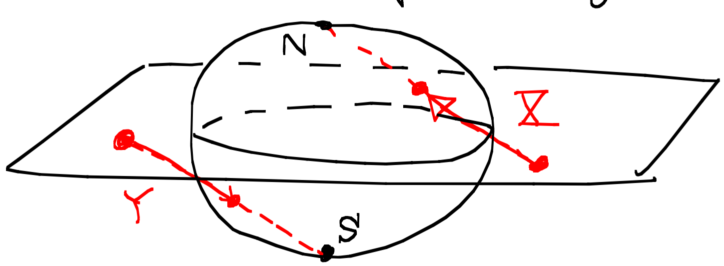
(i.e. X_u, X_v are linearly independent.)

__Def__: A __coordinate patch__ (or __parametrization__) is a __one-to-one__ regular mapping $X : D \to \mathbb{R}^3$ of an open (connected) set $D \subset \mathbb{R}^2$.

Def: A surface in $\mathbb{R}^3$ is a subset $M \subseteq \mathbb{R}^3$ such that each point of M has a neighborhood (in M) contained in the image of some coordinate patch

$$X : D \to M \subset \mathbb{R}^3, \quad D \subset \mathbb{R}^2.$$

eg: $S^2 = \{ (x^1, x^2, x^3) \in \mathbb{R}^3 : \sum_{i=1}^{3} (x^i)^2 = 1 \}$ is a surface: Stereographic projections wrt north N & south S poles give the required coordinate patches.



($X : \mathbb{C} \to S^2$ & $Y : \mathbb{C} \to S^2$ are the required coordinate patches)

Def: A surface M is <u>differentiable</u> (or <u>smooth</u>) if

$\forall$ coordinate patches $\mathbb{X}: D_1 \to M \subset \mathbb{R}^3$ &

$Y: D_2 \to M \subset \mathbb{R}^3$,

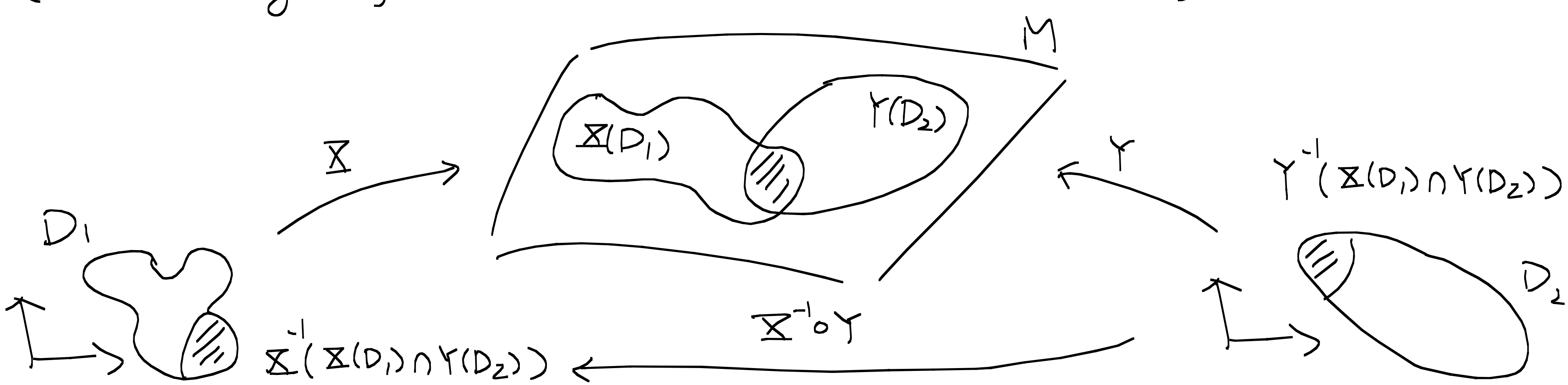
with $\mathbb{X}(D_1) \cap Y(D_2) \neq \phi$,

the composition

$$\underset{}{\mathbb{X}^{-1} \circ Y : \underbrace{Y^{-1}(\mathbb{X}(D_1) \cap Y(D_2))}_{\subset \mathbb{R}^2} \longrightarrow \underbrace{\mathbb{X}^{-1}(\mathbb{X}(D_1) \cap Y(D_2))}_{\subset \mathbb{R}^2}}$$

is differentiable.

(i.e. changes of coordinates are differentiable!)



Note: $\mathbf{X}^{-1} \circ Y$ maps 2 variables to 2 variables:

If
$$\begin{cases} \mathbf{X} : (u_1, v_1) \mapsto (x^1(u_1,v_1), x^2(u_1,v_1), x^3(u_1,v_1)) \\ Y : (u_2, v_2) \mapsto (x^1(u_2,v_2), x^2(u_2,v_2), x^3(u_2,v_2)) \end{cases}$$

Then

$$\mathbf{X}^{-1} \circ Y : (u_2, v_2) \longmapsto (u_1, v_1) \quad \&$$

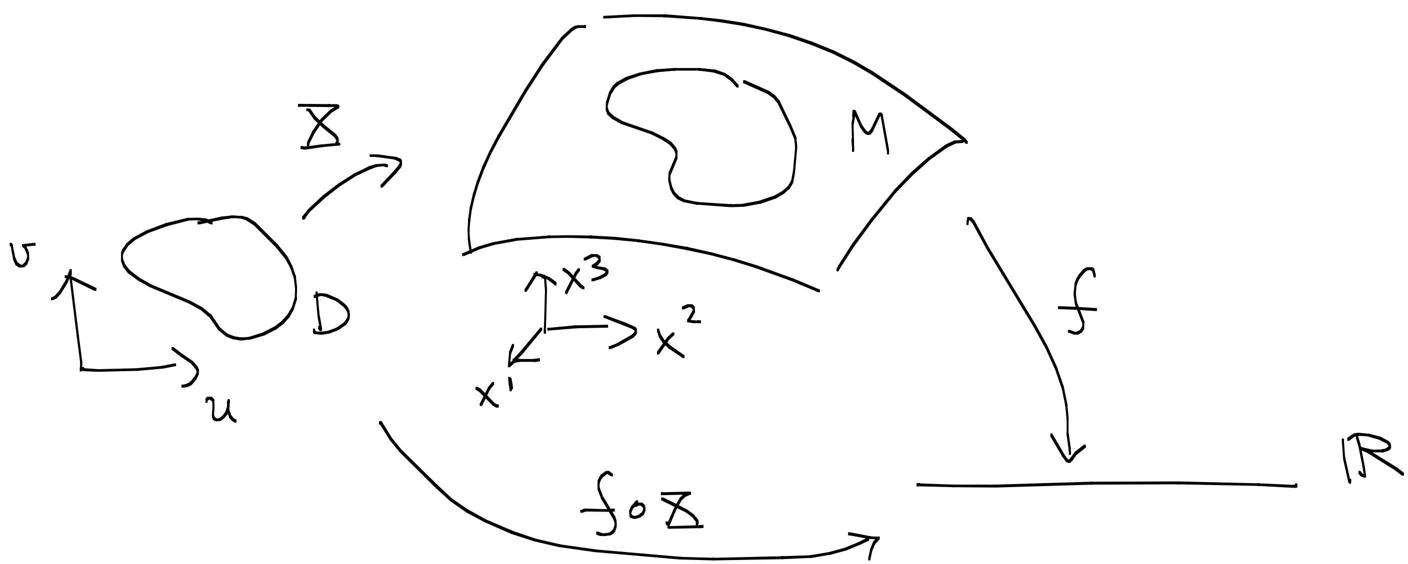
$$Y^{-1} \circ \mathbf{X} : (u_1, v_1) \longmapsto (u_2, v_2)$$

$\therefore$ differentiability for $\mathbf{X}^{-1} \circ Y$ is defined as in Advanced Calculus.

Def: A function $f: M \to \mathbb{R}$ from a surface M is differentiable (or smooth) if

$$f \circ \mathbf{X} : D \to \mathbb{R} \text{ is smooth} \quad \left(\begin{array}{c}\text{coordinate}\\ \text{representation}\end{array}\right)$$

for each coordinate patch $\mathbf{X} : D \to M \subset \mathbb{R}^3$.



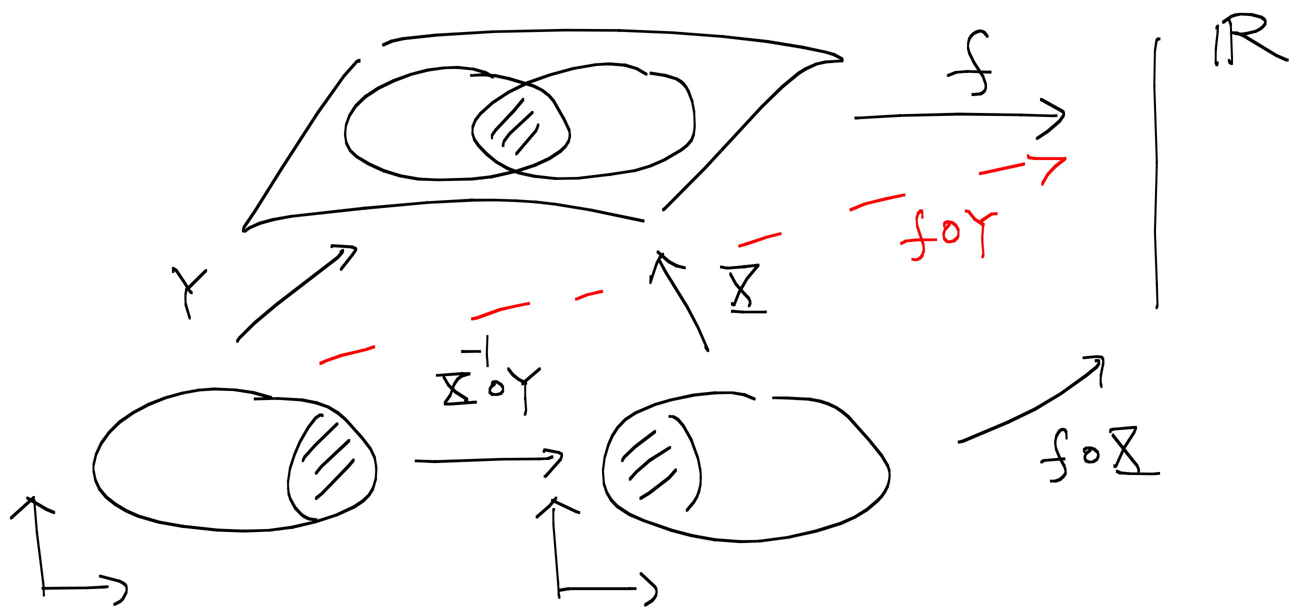
i.e. $f(x^1, x^2, x^3) = f(x^1(u,v), x^2(u,v), x^3(u,v))$

is differentiable wrt u, v. $\left((x^1, x^2, x^3) \in M\right)$

Remark: If $f \circ X$ is smooth for a particular coordinate patch X, then $f \circ Y$ is smooth $\forall$ other coordinate patch overlapping with X:

differentiable

$$f \circ Y = (f \circ X) \circ (X^{-1} \circ Y)$$



Def: A curve on a surface is a mapping (cont.) from an interval to the surface.

eg. $\alpha : [a,b] \to M$ is a curve in M with end points $\alpha(a)$ and $\alpha(b)$.

Def: The surface M is said to be path connected if

$\forall p, q \in M$, $\exists$ (cont.) curve $\alpha : [0,1] \to M$ s.t.

$\alpha(0) = p$ & $\alpha(1) = q$.

- In this course all surfaces, unless explicitly mentioned otherwise, will be assumed to be path connected.

<u>Note</u> : From topology : (in our case)

path connected $\Rightarrow$ connected :

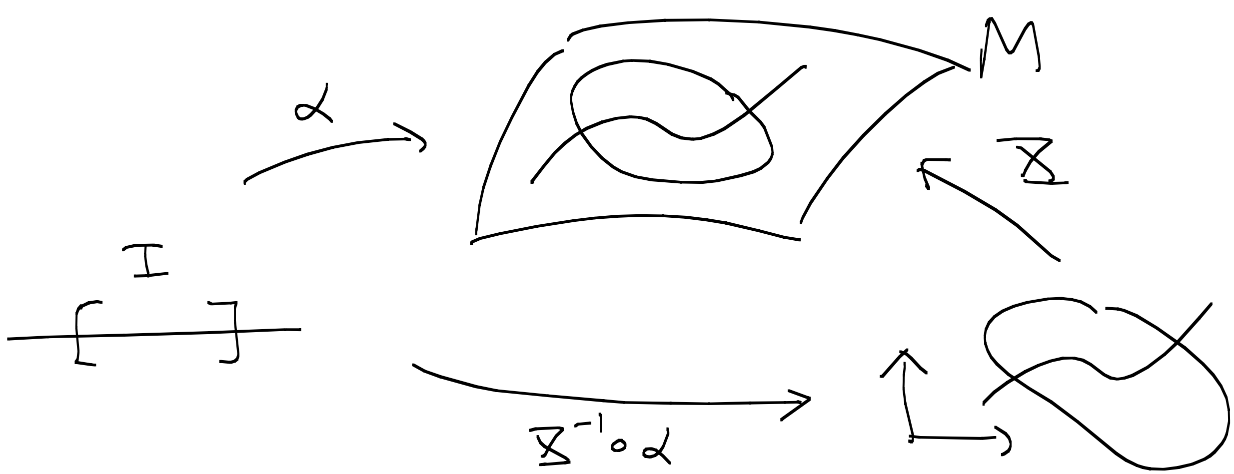
i.e. if $M = A \cup B$ with $A \cap B = \phi$, A & B open in M,

then $A = \phi$ or $B = \phi$.

(i.e. M cannot have 2 (or more) components.)

<u>Def</u> : A curve α is <u>differentiable</u> (or <u>smooth</u>) if

$\mathbf{X}^{-1} \circ \alpha$ is smooth $\forall$ coordinate patch $\mathbf{X}$.

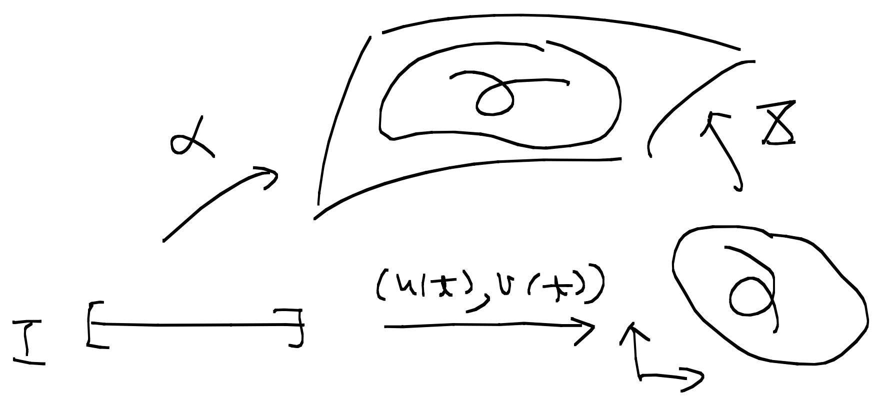


Lemma: ("Important" Lemma 2.1.3 of Oprea)

Let
- M = surface,
- $\mathbf{x}: D \to M$ = (coordinate) patch on M
- $\alpha: I \to \mathbf{x}(D)$ = curve on M contained in $\mathbf{x}(D)$.

Then $\exists$ unique functions $u(t), v(t): I \to \mathbb{R}$ s.t.

$$\boxed{\alpha(t) = \mathbf{x}(u(t), v(t))}.$$



Pf Easy $(u(t), v(t)) = \mathbf{x}^{-1} \circ \alpha(t)$ is the unique choice.

#

<u>Def</u>: Closed curve on surface M are defined as a curve $\alpha : [a, b] \to M$ such that

$$\alpha^{(n)}(a) = \alpha^{(n)}(b) \quad \forall\ n = 0, 1, 2, 3, \cdots$$

(It is equivalent to $\alpha : S^1 \to M$.)

<u>Def</u>: Let • M, N = surfaces,

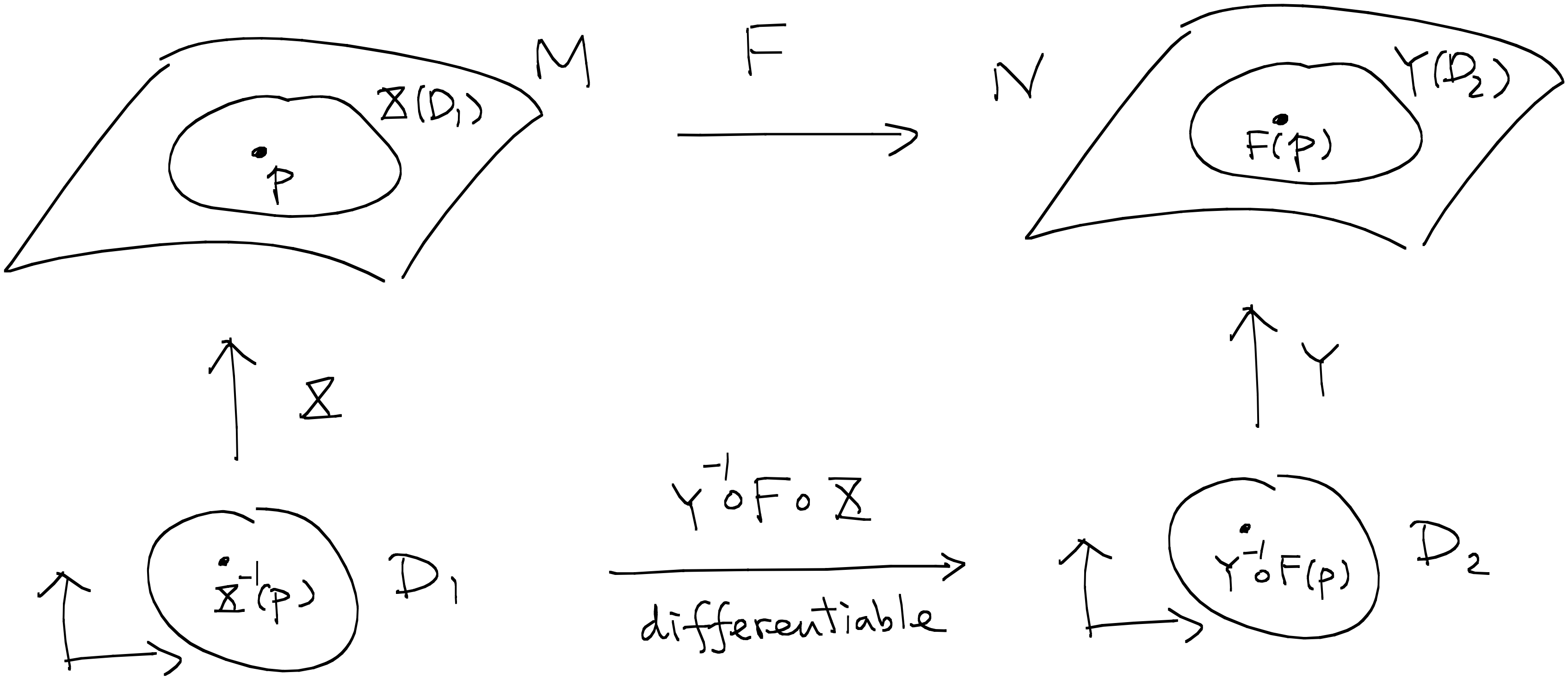
• $F : M \to N$ = mapping from M to N.

Then F is said to be <u>differentiable</u> at a point $p \in M$,

if $\forall$ patches, $\begin{cases} X : D_1 \to M \text{ and} \\ Y : D_2 \to N, \end{cases}$

such that
$$\begin{cases} p \in X(D_1) & \text{and} \\ F(p) \in Y(D_2), & \left(\text{and } F(X_1(D_1)) \subset Y(D_2)\right) \end{cases}$$
we have $Y^{-1} \circ F \circ X$ is differentiable at $X^{-1}(p)$.

(i.e. F is diff $\Leftrightarrow$ its coordinates representation $Y^{-1} \circ F \circ X$ is diff.)



Recall : A surface $M \subset \mathbb{R}^3$ is compact if it is closed and bounded.

Prop :

(1) Continuous image of compact set is compact.

i.e. If M cpt., $F: M \to N$ cts., then $F(M)$ is cpt.

(2) M cpt., $f: M \to \mathbb{R}$ smooth, then f attains its maximum and minimum at some points of M.

Examples of Parametrizations (Patches) on Surfaces.

- **The Monge Patch**

Let $f = f(x,y)$ is a smooth function of 2 variables.

Then its graph $M = \{ z = f(x,y) \} \subset \mathbb{R}^3$ is a surface in $\mathbb{R}^3$.

Check: Let D be the domain of f, then

$$\mathbf{X}(u,v) = (u, v, f(u,v)) : D \to M \subset \mathbb{R}^3$$

is a map from $D \subset \mathbb{R}^2$ to $M \subset \mathbb{R}^3$. $\quad ((x,y,z) = (x^1, x^2, x^3))$

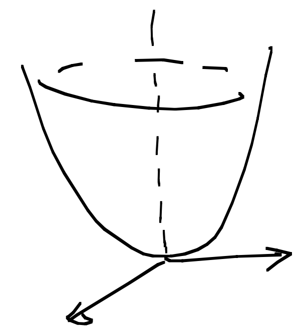
It is clear that $\mathbf{X}$ is 1-1, and is easy to get

$$\begin{cases} \mathbf{X}_u = (1, 0, f_u) \\ \mathbf{X}_v = (0, 1, f_v). \end{cases} \qquad \left(\begin{array}{l} f_u = \frac{\partial f}{\partial u} \\ f_v = \frac{\partial f}{\partial v} \end{array} \right)$$

$$\Rightarrow \quad X_u \times X_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix}$$

$$= (-f_u, -f_v, 1) \neq 0 \quad \forall (u,v) \in D$$

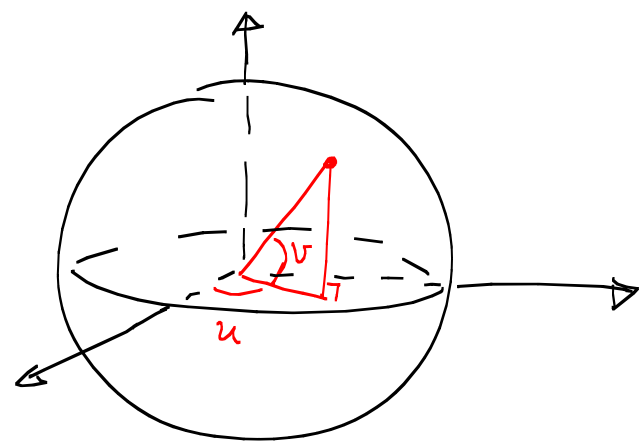
$\therefore$ X is <u>regular</u>.



For explicit example, take $f(x,y) = x^2 + y^2$.

Then $X(u,v) = (u, v, u^2+v^2)$ (non-compact surface.)

- <u>Geographical Coordinates of a Sphere</u>



(induced from polar coordinates on $\mathbb{R}^3$)

Let S^2_R = sphere of radius R

Then for $0 < u < 2\pi$, $-\pi/2 < v < \pi/2$

$\underline{X}(u,v) = (R\cos u\cos v,\ R\sin u\cos v,\ R\sin v) \in S^2_R$

And $\begin{cases} \underline{X}_u = (-R\sin u\cos v,\ R\cos u\cos v,\ 0) \\ \underline{X}_v = (-R\cos u\sin v,\ -R\sin u\sin v,\ R\cos v) \end{cases}$

$\Rightarrow \quad \underline{X}_u \times \underline{X}_v = (R^2\cos u\cos^2 v,\ R^2\sin u\cos^2 v,\ R^2\sin v\cos v)$

$\therefore \quad |\underline{X}_u \times \underline{X}_v|^2 = R^4\cos^2 v > 0 \quad$ as $-\pi/2 < v < \pi/2$

$\therefore \quad \underline{X}$ is a regular patch on S^2_R. (not the whole S^2_R.)

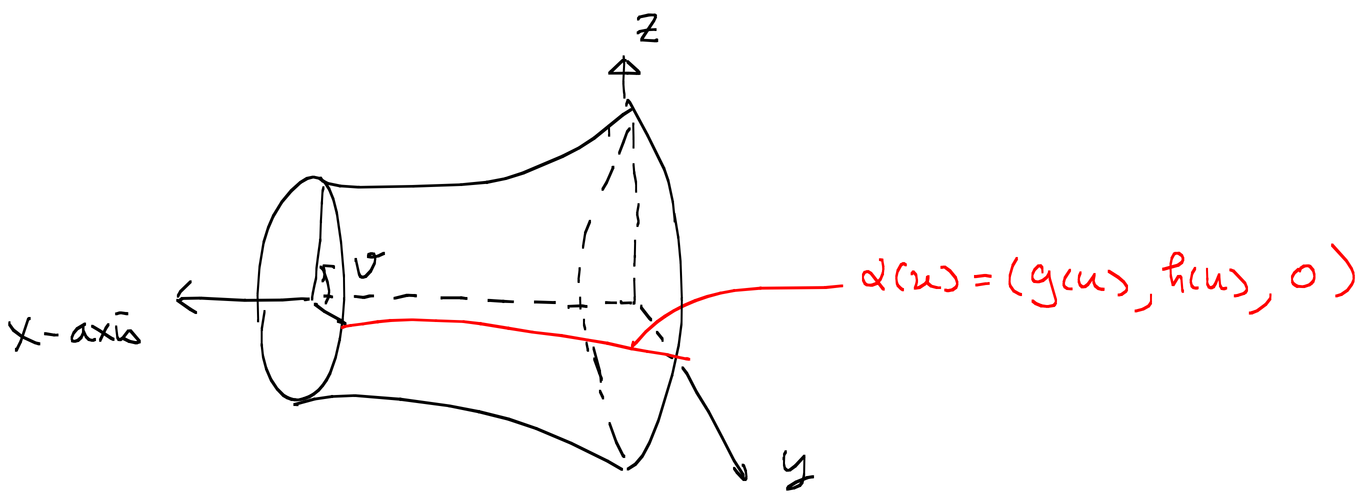
• Surfaces of Revolution

Let $\alpha(u) = (g(u), h(u), 0)$ be a plane curve in xy-plane.

Revolve α about the x-axis

one obtains a surface of revolution parametrized by

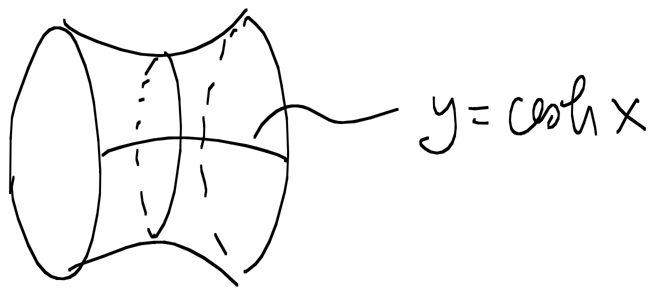
$$X(u,v) = (g(u),\ h(u)\cos v,\ h(u)\sin v). \qquad (0 < v < 2\pi)$$



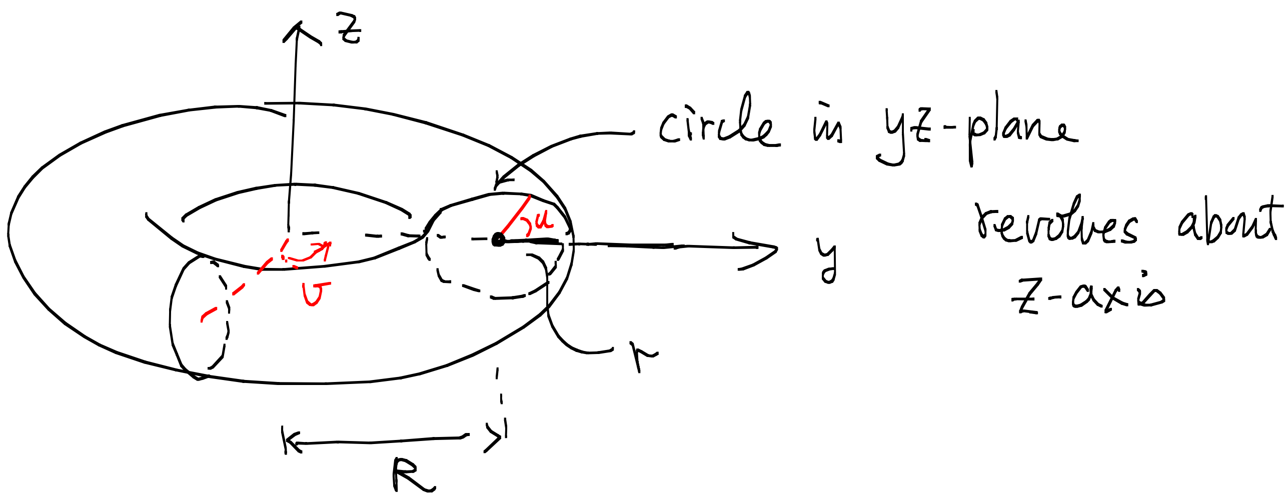
Exercise: check $X_u \times X_v = (hh', -hg'\cos v, -hg'\sin v)$

and $X_u \times X_v \neq 0$ if α is regular and has no intersection with x-axis.

eg: The curve $y = \cosh x$ (catenary) revolves about x-axis gives the catenoid:



eg Torus.



$$\mathbf{x}(u,v) = \big((R + r\cos u)\cos v,\ (R + r\cos u)\sin v,\ r\sin u \big)$$

is a parametrization on the torus. (not the whole torus. what are missing?)

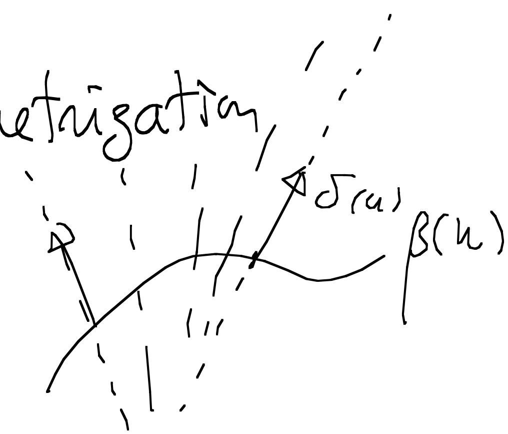
(Ex. 2.1.13 of Oprea)

- **Ruled surfaces:**

Def A surface is ruled if it has a parametrization

$$X(u,v) = \beta(u) + v\,\delta(u)$$

where β, δ are curves.



Note:
- $\forall$ fixed $u = u_0$, $X(u_0, v) = \beta(u_0) + v\,\delta(u_0)$ is a straight line (passing thro. $\beta(u_0)$ in the direction $\delta(u_0)$) completely contained in the surface.

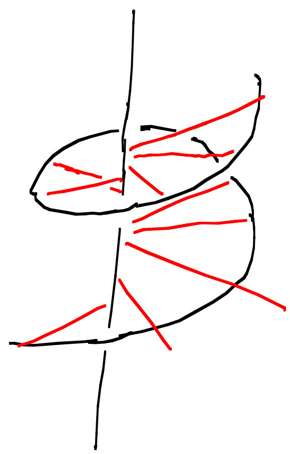
- The ruled surface is covered by this family of straight lines. (This is the reason for the name!)

- $\beta(u)$ is called the directrix, & $\delta(u)$ is called a ruling of the surface.

eg <u>Helicoid</u>

Take a helix $\alpha(u) = (a\cos u, a\sin u, bu)$,

Then draw a line through $(0, 0, bu)$ and $(a\cos u, a\sin u, bu)$ $(\forall u)$. The surface swept out by these lines is a helicoid.



$\therefore$ $X(u,v) = (av\cos u, av\sin u, bu)$ is a parametrization on the helicoid.

(Ex: Check that it is regular.)

Since

$$X(u,v) = (0, 0, bu) + v(a\cos u, a\sin u, 0)$$
$$= \beta(u) + v\delta(u),$$

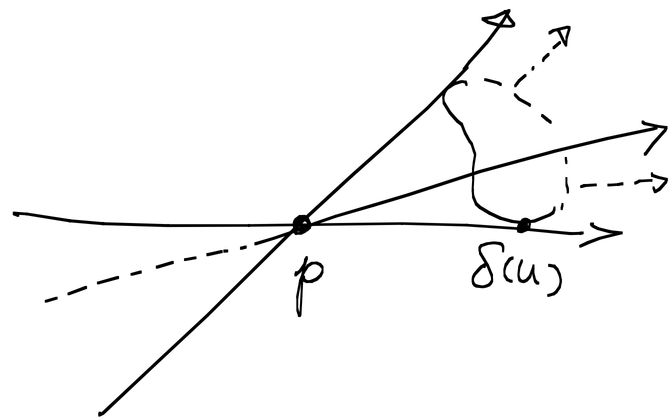
the Helicoid is a ruled surface.

eg. A Cone is ruled:

$$\mathbf{x}(u,v) = p + v\,\delta(u)$$

$$(\beta(u) = p = \text{const.})$$

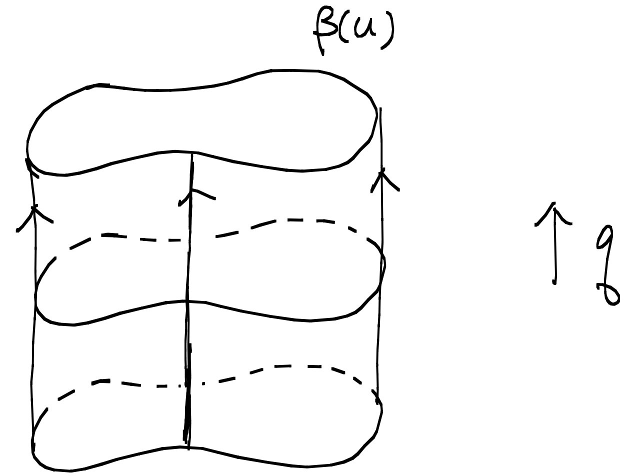
(Cone over the curve $\delta(u)$)



eg. Cylinders are ruled

$$\mathbf{x}(u,v) = \beta(u) + v\,q$$

$$(\delta(u) = q = \text{const.})$$



2.2 Geometry of Surfaces

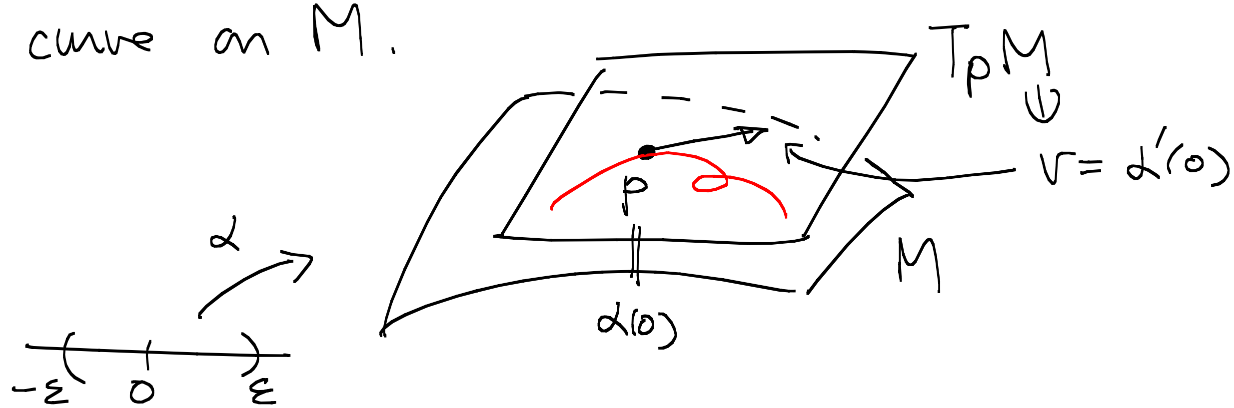
<u>Def</u>: • The <u>tangent plane</u> $T_p(M)$ of M at p is defined to be

$$T_p(M) = \left\{ \alpha'(0) : \begin{array}{c} \alpha : (-\varepsilon, \varepsilon) \to M \\ \alpha(0) = p \end{array}, \text{ for some } \varepsilon > 0 \right\}$$

(For simplicity, we will also use T_pM for $T_p(M)$.)

• Elements in $T_p(M)$ are called <u>tangent vectors</u> to M at p.

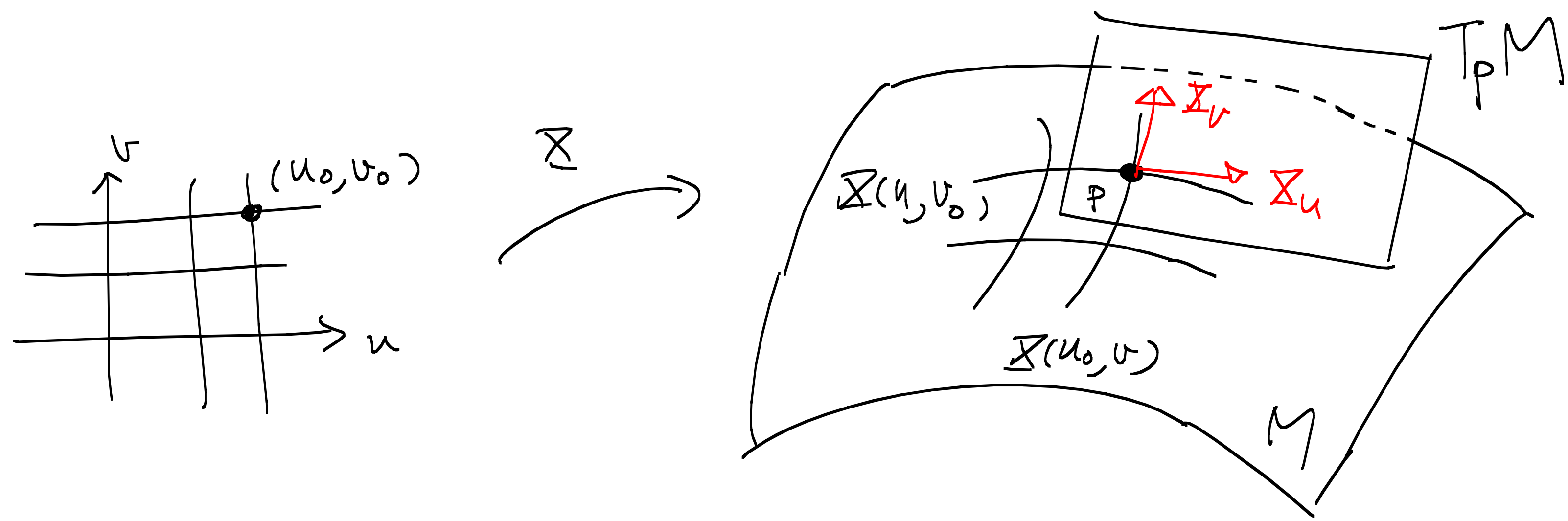
i.e. a tangent vector of M at p is the velocity vector (at p) of some curve on M.



eg: If $\mathbf{X}(u,v)$ is a coordinate patch of M, then

at $p = \mathbf{X}(u_0, v_0)$, $\mathbf{X}_u$ & $\mathbf{X}_v \in T_pM$:

they are velocity vectors of the u-parameter curve $\mathbf{X}(u, v_0)$ and v-parameter curve $\mathbf{X}(u_0, v)$ at p respectively.



Lemma (Lemma 2.2.1 of Oprea)

Let

- M = regular surface,
- $p \in M$,
- $\mathbf{x}(u,v)$ = regular patch on M s.t. $p = \mathbf{x}(u_0, v_0)$

Then T_pM is a vector space with basis $\{\mathbf{x}_u, \mathbf{x}_v\}_{(u_0,v_0)}$.

Pf: Step 1: $\mathbf{X}_u, \mathbf{X}_v$ are linearly independent.

Pf: $\mathbf{X}$ regular $\Rightarrow$ $\mathbf{X}_u \times \mathbf{X}_v \neq 0$

$\Rightarrow$ $\mathbf{X}_u, \mathbf{X}_v$ are linearly indep.

Step 2: $T_pM \subset \operatorname{span}\{\mathbf{X}_u, \mathbf{X}_v\}_{(u_0, v_0)}$.

Pf: $\forall\, v \in T_pM$, $\exists$ a curve $\alpha : (-\varepsilon, \varepsilon) \to M$

such that $\begin{cases} \alpha(0) = p & \& \\ \alpha'(0) = v \end{cases}$

By the "important" lemma, $\exists$ functions

$u(t)$ & $v(t)$ on $(-\varepsilon, \varepsilon)$ such that

$$\begin{cases} \alpha(t) = X(u(t), v(t)), & \forall t \in (-\varepsilon, \varepsilon). \\ \& \ (u(0), v(0)) = (u_0, v_0). \end{cases}$$

Hence $V = \alpha'(0) = X_u(u_0,v_0)\, u'(0) + X_v(u_0,v_0)\, v'(0)$

$$\in \operatorname{span}\{X_u, X_v\}_{(u_0,v_0)}$$

Since $V \in T_pM$ is arbitrary, we have

$$T_pM \subset \operatorname{span}\{X_u, X_v\}_{(u_0,v_0)}.$$

Step 3: $T_pM = \operatorname{span}\{X_u, X_v\}_{(u_0,v_0)}$ and hence T_pM is a vector space with basis $\{X_u, X_v\}_{(u_0,v_0)}$.

Pf: By Steps 1 & 2, we only need to show that $\operatorname{span}\{X_u, X_v\}_{(u_0,v_0)} \subset T_pM$.

Suppose $V \in \operatorname{span}\{X_u, X_v\}_{(u_0,v_0)}$.

Then $\exists$ constants λ, μ s.t.

$$v = \lambda X_u + \mu X_v \quad (\text{at } (u_0, v_0))$$

Define a curve

$$\alpha(t) = X(u_0 + t\lambda, v_0 + t\mu)$$

for $t \in (-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ small enough.

($\varepsilon > 0$ small enough $\Rightarrow$ $(u_0 + t\lambda, v_0 + t\mu)$ belongs to the domain of the patch X.)

Then $\alpha(0) = X(u_0, v_0) = p$

$$\alpha'(0) = \lambda X_u + \mu X_v = v$$

$\therefore \quad v \in T_p M$.

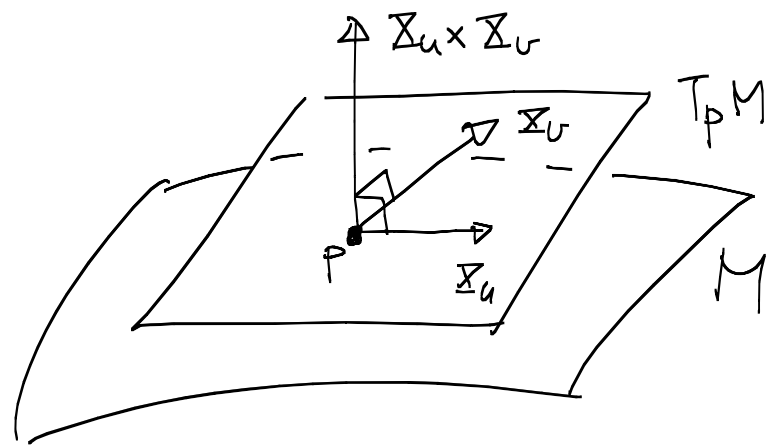
This completes the proof of the lemma. #

Cor : Let $\mathbb{X}(u,v)$ = regular coordinate patch,

then
$$U = \frac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|}$$
is a unit normal

to the tangent plane T_pM, where $p = \mathbb{X}(u,v)$.

(Pf : Easy)



eg : M = graph of "$z = f(x,y)$"

Monge patch : $\mathbb{X}(u,v) = (u, v, f(u,v))$

$$\Rightarrow \begin{cases} \mathbb{X}_u = (1, 0, f_u) \\ \mathbb{X}_v = (0, 1, f_v) \end{cases}$$

$$\Rightarrow \quad \mathbb{X}_u \times \mathbb{X}_v = (-f_u, -f_v, 1)$$

is normal to the tangent plane (not <u>unit</u>)

$\therefore$ $\forall$ point $(x, y, z) \in T_pM$, where $p = (u_0, v_0, f(u_0, v_0))$

we have

$$\langle (x, y, z) - p, \ \mathbb{X}_u \times \mathbb{X}_v \big|_{(u_0, v_0)} \rangle = 0$$

i.e. $\quad -f_u(u_0, v_0)(x - u_0) - f_v(u_0, v_0)(y - v_0) + (z - f(u_0, v_0)) = 0$

is the eqt. of T_pM.

<u>Explicit eg</u>: If $f(x,y) = x^2 + y^2$ (Paraboloid)

Then $\quad f_u = 2u, \quad f_v = 2v$

$\Rightarrow \quad T_p(\{z = x^2 + y^2\})$ is given by the equation

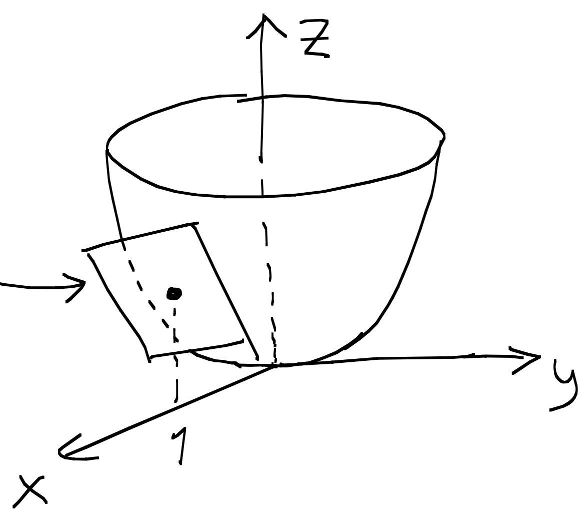
$$-2u_0(x - u_0) - 2v_0(y - v_0) + (z - (u_0^2 + v_0^2)) = 0$$

i.e. $\qquad 2u_0 x + 2v_0 y - z = u_0^2 + v_0^2$

Even more explicitly, if $p = (\underset{u_0}{\underset{\parallel}{1}}, \underset{v_0}{\underset{\parallel}{0}}, \underset{u_0^2 + v_0^2}{\underset{\parallel}{1}})$, then

the tangent plane is

$$2x - z = 1$$



eg. Sphere of radius 1 : $S^2 = \{ x^2+y^2+z^2 = 1 \}$

Recall the parametrization

$$X(u,v) = (\cos u \cos v, \sin u \cos v, \sin v) \qquad (0 < u < 2\pi)$$

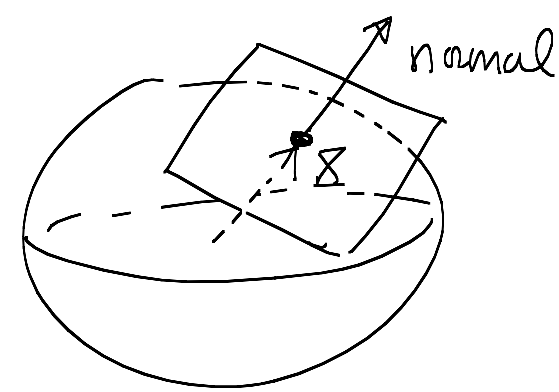
$$\left(-\frac{\pi}{2} < v < \frac{\pi}{2} \right)$$

which has a normal

$$X_u \times X_v = (\cos u \cos^2 v, \sin u \cos^2 v, \sin v \cos v)$$

$$= \cos v \, X \qquad (\cos v > 0 \text{ for } -\pi/2 < v < \pi/2)$$

i.e. the normal & the position vector X are in the same radial direction.



Hence, the equation of the tangent plane at $p (= \mathbb{X})$

is $\quad \langle (x,y,z) - p, p \rangle = 0$

on S^2

i.e. $\quad \langle (x,y,z), p \rangle = \langle p, p \rangle = 1$

If $p = (x_0, y_0, z_0)$, then the tangent plane $T_p S^2$ is

$$x_0 x + y_0 y + z_0 z = 1 .$$

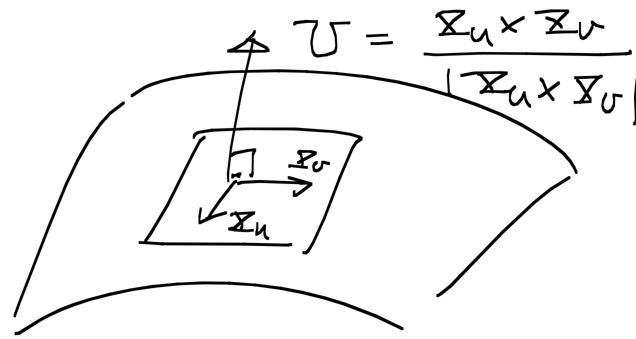
<u>Orientable Surfaces</u>:

<u>Def</u>: A regular surface $M \subset \mathbb{R}^3$ is orientable

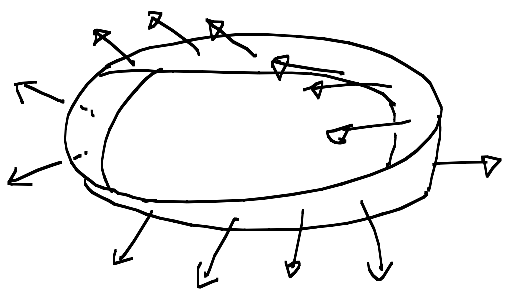
$\Longleftrightarrow$ $\exists$ a differentiable field of <u>unit normal</u> vectors $U : M \to S^2 \subset \mathbb{R}^3$ (Gauss map)

<u>eg</u>: A (single) regular patch $\mathbb{X} : D \to M = \mathbb{X}(D)$ is

always orientable : $\nu = \dfrac{X_u \times X_v}{|X_u \times X_v|}$ is the required unit normal vector field.



<u>eg</u>: Möbius strip is <u>not</u> orientable :



different direction
$\Rightarrow$ discontinuity.

(See eg 2.2.4 of Oprea for details.)

Note : Except a few exceptions, all surfaces in this course are orientable.

Directional derivative

Def: Let M = regular surface in $\mathbb{R}^3$,

g = function on M,

$V \in T_pM$, $(p \in M)$

Then the <u>directional derivative</u> of g in the <u>V-direction</u>

is
$$V[g](p) \overset{\text{def}}{=\!=} \left.\frac{d}{dt}\right|_{t=0} g(\alpha(t))$$

where $\alpha(t)$ is any curve on M with $\alpha(0)=p$, $\alpha'(0)=V$.

<u>Notes</u>: • It is easy to see by chain rule that if g is in fact a restriction of some function defined in $\mathbb{R}^3$, then

$$\left.\frac{d}{dt}\right|_{t=0} g(\alpha(t)) = \left.\frac{d}{dt}\right|_{t=0} g(\alpha^1(t), \alpha^2(t), \alpha^3(t))$$

$$= \frac{\partial g}{\partial x^1}\frac{d\alpha^1}{dt} + \frac{\partial g}{\partial x^2}\frac{d\alpha^2}{dt} + \frac{\partial g}{\partial x^3}\frac{d\alpha^3}{dt}\Big|_{t=0}$$

$$= \langle \nabla g(p), v \rangle$$

where $\nabla g(p) = \left(\frac{\partial g}{\partial x^1}(p), \frac{\partial g}{\partial x^2}(p), \frac{\partial g}{\partial x^3}(p)\right) \in \mathbb{R}^3$

is the gradient of g.

$$\therefore \quad \boxed{v[g](p) = \langle \nabla g(p), v \rangle}$$

Therefore, $v[g](p)$ is independent of the particular curve α. (only on $\alpha(0) = p$ & $\alpha'(0) = v$) (Note: any function on a small nbd of M can be extended to $\mathbb{R}^3$.)

- Directional derivative can be regarded as an <u>action</u> of $v \in T_pM$ on space of smooth function $g : M \to \mathbb{R}$ (to produce a scalar $v[g](p)$)

- If we have a <u>vector field</u> $V_p \in T_pM$, $\forall\, p \in M$. Then we can consider the function

$$p \longmapsto V_p[g](p) \quad \left(\text{denoted by } V_p[g]\right).$$

In this case, vector field acts on functions to produce functions.

<u>Notation</u>: Suppose $\mathbf{X} : D \to M \subset \mathbb{R}^3$ is a coordinate patch.

$$(u,v) \mapsto \mathbf{X}(u,v)$$

For any function f on M, we denote

$$\begin{cases} \dfrac{\partial f}{\partial u} \overset{\text{notation}}{=\!=} \dfrac{\partial}{\partial u}(f \circ \mathbf{X}) \\[2ex] \dfrac{\partial f}{\partial v} \overset{\text{notation}}{=\!=} \dfrac{\partial}{\partial v}(f \circ \mathbf{X}) \end{cases} \quad \left(\text{or simply } \begin{matrix} f_u \\[2ex] f_v \end{matrix}\right)$$

Then at a point $p = \mathbf{X}(u_0, v_0)$, we have

$$\mathbf{X}_u[f](p) = \left.\frac{d}{du}\right|_{u=u_0} f(\mathbf{X}(u, v_0)) \qquad (\text{why?})$$

$$= \left.\frac{\partial(f\circ\mathbf{X})}{\partial u}(u, v_0)\right|_{u=u_0}$$

$$= \frac{\partial f}{\partial u}(u_0, v_0).$$

Similarly $\quad \mathbf{X}_v[f](p) = \dfrac{\partial f}{\partial v}(u_0, v_0)$

[This is why we use notation

$\dfrac{\partial}{\partial u}, \dfrac{\partial}{\partial v}$ or ∂_u, ∂_v for $\mathbf{X}_u, \mathbf{X}_v$

in more advanced level differential geometry.]

<u>Note</u>: In case that f is a restriction of a function on $\mathbb{R}^3$, the notation is consistent with the chain rule

$$\boxed{\frac{\partial f}{\partial u} = \mathbf{x}_u[f] = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial u}}$$

since

$$\mathbf{x}_u[f] = \langle \nabla f, \mathbf{x}_u \rangle \quad \& \quad \mathbf{x}_u = \left(\frac{\partial x^1}{\partial u}, \frac{\partial x^2}{\partial u}, \frac{\partial x^3}{\partial u}\right).$$

<u>eg</u> (exercise 2.2.7 of Oprea, note the misprint in Oprea)

$$\boxed{\mathbf{x}_v[\mathbf{x}_u[f]] = \sum_j \sum_i \frac{\partial^2 f}{\partial x^i \partial x^j}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v} + \sum_i \frac{\partial f}{\partial x^i}\frac{\partial^2 x^i}{\partial v \partial u}.}$$

(Pf: Ex)

Covariant derivative of the Gauss map:

Recall: The Gauss map is a unit normal vector field on M,

i.e. $U : M \to \mathbb{R}^3$ with $|U| = 1$

In the case of a coordinate patch $X(u,v)$,

the Gauss map (compatible with the orientation)

is

$$U = \frac{X_u \times X_v}{|X_u \times X_v|} .$$

If we write $\quad U = u^1 e_1 + u^2 e_2 + u^3 e_3$

where $\quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \& \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

Then u^1, u^2, u^3 are functions on M.

For any $v \in T_pM$, we denote

$$\boxed{\nabla_v U = \sum_{i=1}^{3} v[u^i] e_i}$$

(More precisely, we should write $\nabla^{\mathbb{R}^3}_v U$ instead because this is really the $\mathbb{R}^3$-covariant derivative of U as defined.)

Remarks: For a curve $\alpha(t)$ on M, we write

$$\nabla_{\alpha'(t)} U = \sum_{i=1}^{3} \frac{d}{dt} u^i(\alpha(t))\, e_i$$

(may simply denoted by U_t if the curve is clear.)

<u>Def</u>: The <u>Shape Operator</u> (or <u>Weingarten Map</u>) of a surface M at p is defined by

$$\boxed{S_p(v) = -\nabla_v U, \quad v \in T_pM,}$$

where U is the Gauss map of M.

Lemma (Lemma 2.2.10 of Oprea)

S_p is a linear transformation from T_pM onto itself.

Pf: Since $|U|=1$,

$$0 = v[\langle U, U\rangle] = v\Big[\sum_i (u^i)^2\Big]$$

(check) $$= 2\sum_i u^i\, v[u^i] = 2\langle U, \nabla_v U\rangle$$

$$\Rightarrow \nabla_v U \in T_pM. \quad (U = \text{normal to } T_pM)$$

$\therefore$ S_p is a mapping from T_pM into itself.

To see S_p is linear, we observe that by the definition of directional derivatives (ex. 2.2.11 of Oprea), we have

$$(av+w)[u^i] = a\,v[u^i] + w[u^i], \quad \forall i = 1,2,3$$

$\forall$ any vectors $v, w \in T_pM$ & scalar a.

Therefore
$$S_p(av+w) = -\nabla_{av+w} U$$
$$= -(a\nabla_v U + \nabla_w U)$$
$$= aS_p(v) + S_p(w).$$

$\therefore$ S_p is linear. $\divideontimes$

eg M = plane in $\mathbb{R}^3$

Then U = constant vector $\left(\text{i.e. } \sum_{i=1}^{3} u^i e_i,\quad u^i = \text{const.}\right)$

$\Rightarrow \nabla_v U = 0 \quad \forall\, v \in T_pM$

$\therefore$ $S_p = 0$.

eg: $S^2(R)$ = sphere of radius R in $\mathbb{R}^3$ $(= S^2_R)$

Consider the parametrization (covers most of the S^2_R)

$$\mathbf{X}(u,v) = (R\cos u\cos v,\ R\sin u\cos v,\ R\sin v).$$

Then
$$\begin{cases} \mathbf{X}_u = R(-\sin u\cos v,\ \cos u\cos v,\ 0) \\ \mathbf{X}_v = R(-\cos u\sin v,\ -\sin u\sin v,\ \cos v) \end{cases}$$

$$\Rightarrow \quad \mathbf{X}_u \times \mathbf{X}_v = R^2\cos v\,(\cos u\cos v,\ \sin u\cos v,\ \sin v)$$

$$\Rightarrow \quad U = (\cos u\cos v,\ \sin u\cos v,\ \sin v) \quad \left(= \frac{\mathbf{X}(u,v)}{R}\right)$$

$$\therefore\ S_p(\mathbf{X}_u) = -\nabla_{\mathbf{X}_u} U = -\sum_i \mathbf{X}_u[u^i]e_i$$

$$= -\sum_i \frac{\partial}{\partial u}(u^i)\, e_i$$

$$= -(-\sin u\cos v,\ \cos u\cos v,\ 0)$$

$$= -\frac{1}{R} X_u$$

& $$S_p(X_v) = -\nabla_{X_v} U$$

$$= -(-\cos u \sin v, -\sin u \sin v, \cos v)$$

$$= -\frac{1}{R} X_v$$

Therefore $\forall\ v = \lambda X_u + \mu X_v \in T_pM$,

$$S_p(v) = -\frac{1}{R}(\lambda X_u + \mu X_v) = -\frac{1}{R} v$$

i.e. $$S_p = -\frac{1}{R} \cdot Id_{T_pM}$$

(i.e. $S_p = -\frac{1}{R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ wrt a basis.)

Thm (Thm 2.2.17 of Oprea)

If $S_p = 0$ $\forall p \in M$, then M is contained in a plane.

(Recall that all surfaces are path connected in this course.)

Pf: Fix a point $p \in M$ with unit normal $U(p)$.

Then $\forall q \in M$, $\exists$ curve $\alpha(t)$, $0 \le t \le 1$, on M s.t.

$$\alpha(0) = p \quad \& \quad \alpha(1) = q.$$

Consider the function

$$f(t) = \langle \alpha(t) - q, U(\alpha(t)) \rangle.$$

Then
$$f'(t) = \langle \underset{\substack{\uparrow \\ \text{tangent}}}{\alpha'(t)}, \underset{\substack{\uparrow \\ \text{normal}}}{U(\alpha(t))} \rangle + \langle \alpha(t) - q, \underset{\substack{\| \\ -S_{\alpha(t)}(\alpha'(t)) = 0}}{\nabla_{\alpha'(t)} U} \rangle$$

$\therefore \quad f'(t) = 0 \quad \forall\, t \in [0,1]$ and hence f is a constant.

$$\therefore \quad \langle p-q, U(p) \rangle = f(0) = f(1) = \langle q-q, U(q) \rangle = 0$$

$\Rightarrow \quad \forall q \in M, \quad q \in$ plane passing thro. p with normal $U(p)$.

$\therefore \quad M \subset$ a plane. $\#$

[This justifies the term "Shape Operator" as S_p measures the different of a surface, in shape, from being a plane.]

2.3 The Linear Algebra of Surfaces

Recall: • For $T: V \to V$ linear, and a basis $\mathcal{B} = \{ X_1, \cdots, X_n \}$ of V,

T may be represented by a matrix A (matrix representation of T wrt basis $\mathcal{B}$):

$$T(X_j) = \sum_{i=1}^{n} a^i_j X_i \iff A = (a^i_j)$$

row

column

[Caution: Oprea uses:

$$T(X_i) = \sum_{j=1}^{n} a^{ji} X_j \iff A = (a^{ji}).$$]

• $v \neq 0 \in V$ is an <u>eigenvector</u> of T with <u>eigenvalue</u> λ if $\boxed{Tv = \lambda v}$.

- T is called diagonalizable if $\exists$ a basis for V consists of eigenvectors $v_1, \cdots, v_n$. In this case, if $\lambda_1, \cdots, \lambda_n$ are the corresponding eigenvalues, then the matrix representation of T in the basis $\{v_1, \cdots, v_n\}$ is

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

- $\det(a^i_j) = \det(T) = \prod_{i=1}^{n} \lambda_i$

  $\sum_{i=1}^{n} a^i_i = tr(T) = \sum_{i=1}^{n} \lambda_i$

- If $a^i_j = a^j_i \quad \forall i, j = 1, \cdots, n$, then $A = (a^i_j)$ is called <u>symmetric</u>.

- If $A = \begin{pmatrix} | & & | \\ X_1 & \cdots & X_n \\ | & & | \end{pmatrix}$ with $\langle X_i, X_j \rangle = \delta_{ij}$,

  then A is called <u>orthogonal</u>. In this case, $\{X_1, \cdots, X_n\}$ is called an <u>orthonormal</u> basis (for $\mathbb{R}^n$).

- A linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is called <u>symmetric</u> if $\langle Tv, w \rangle = \langle v, Tw \rangle$, $\forall v, w \in \mathbb{R}^2$.

- If $\mathcal{B} = \{X_1, X_2\}$ is an <u>orthonormal basis</u>, then the matrix representation of a symmetric operator wrt $\mathcal{B}$ is symmetric. (Pf: Exercise!)

<u>Thm</u> (Thm 2.3.5 of Oprea)

The shape operator $S_p: T_pM \to T_pM$ (of a regular surface M at a point p) is a <u>symmetric</u> linear operator.

Furthermore, wrt a coordinate patch (containing p)

$X : (u,v) \longmapsto M$, we have

$$\langle S_p(X_v), X_u\rangle = \boxed{\begin{aligned} \langle S_p(X_u), X_u\rangle &= \langle X_{uu}, U\rangle \\ \langle S_p(X_u), X_v\rangle &= \langle X_{uv}, U\rangle \\ \langle S_p(X_v), X_v\rangle &= \langle X_{vv}, U\rangle, \end{aligned}}$$

where $U = \dfrac{X_u \times X_v}{|X_u \times X_v|}$ is the Gauss map of M at p,

$$X_{uv} = \left(\frac{\partial^2 x^1}{\partial u \partial v}, \frac{\partial^2 x^2}{\partial u \partial v}, \frac{\partial^2 x^3}{\partial u \partial v}\right) \text{ \& etc.}$$

<u>Pf</u>: Since $\{X_u, X_v\}$ is a basis for T_pM, in order to show that S_p is symmetric (i.e. $\langle S_p(v), w\rangle = \langle v, S_p(w)\rangle \ \forall v, w \in T_pM$), we only need to check

$$\langle S_p(X_u), X_v\rangle = \langle S_p(X_v), X_u\rangle.$$

To prove this, we note that

$$\langle \mathbf{X}_u, U \rangle = 0 = \langle \mathbf{X}_v, U \rangle \quad \left(\begin{array}{l} U \text{ normal} \\ \mathbf{X}_u, \mathbf{X}_v \text{ tangent} \end{array} \right)$$

$$\Rightarrow \quad 0 = \mathbf{X}_u \langle \mathbf{X}_v, U \rangle$$

$$= \mathbf{X}_u \left[\sum_{i=1}^{3} u^i \frac{\partial x^i}{\partial v} \right], \quad \text{where } U = \sum_{i=1}^{3} u^i e_i$$

$$= \sum_{i=1}^{3} \left(\mathbf{X}_u [u^i] \frac{\partial x^i}{\partial v} + u^i \, \mathbf{X}_u \left[\frac{\partial x^i}{\partial v} \right] \right)$$

$$= \langle \nabla_{\mathbf{X}_u} U, \mathbf{X}_v \rangle + \langle U, \mathbf{X}_{vu} \rangle$$

$$\therefore \quad \langle S_p(\mathbf{X}_u), \mathbf{X}_v \rangle = \langle -\nabla_{\mathbf{X}_u} U, \mathbf{X}_v \rangle$$

$$= \langle \mathbf{X}_{vu}, U \rangle.$$

Interchanging the role of $\mathbf{X}_u$ & $\mathbf{X}_v$, we have similarly

$$\langle S_p(\mathbf{X}_v), \mathbf{X}_u \rangle = \langle \mathbf{X}_{uv}, U \rangle.$$

Since $\mathbf{X}(u,v)$ is smooth (at least C^2), we have $\mathbf{X}_{uv} = \mathbf{X}_{vu}$.

$\therefore \quad \langle S_p(X_v), X_u \rangle = \langle S_p(X_u), X_v \rangle = \langle X_{uv}, U \rangle$.

This shows that $S_p : T_pM \to T_pM$ is symmetric and the 2^{nd} formula of the theorem.

The other 2 formula can be proved similarly. ※

<u>Cor</u> (cor 2.3.6 of Oprea)

$S_p : T_pM \to T_pM$ has real eigenvalues and diagonalizable.

(Pf : Theory of linear algebra.)

<u>Matrix representation</u> of $S_p : T_pM \to T_pM$ wrt basis $\{X_u, X_v\}$.

(Note that $\{X_u, X_v\}$ is <u>not</u> orthonormal, the matrix may not be symmetric even S_p is symmetric.)

Let us denote $\{X_u, X_v\}$ by $\{X_1, X_2\}$ and let

$$S_p(X_j) = \sum_{i=1}^{2} a_j^i X_i$$

i.e. $A = (a_j^i)$ is the matrix representing S_p wrt $\{X_1, X_2\}$.

<u>Modern notation</u>:

By above $\langle S_p(X_i), X_j \rangle = \langle X_{ij}, U \rangle$

$$\Rightarrow \quad \sum_{k=1}^{2} a_i^k \langle X_k, X_j \rangle = \langle X_{ij}, U \rangle$$

Let $g_{ij} = \langle X_i, X_j \rangle$ & $h_{ij} = \langle X_{ij}, U \rangle$, $\forall\, i,j = 1,2$

Then $$\sum_{k=1}^{2} g_{jk}\, a_i^k = h_{ij} = h_{ji}$$

If we set

$$\mathcal{G} = (g_{ij})_{2\times 2} \ \& \ \mathcal{H} = (h_{ij})_{2\times 2}$$ ($\mathcal{G}, \mathcal{H}$ are 2×2 symmetric matrices.)

then $$\mathcal{G} A = \mathcal{H}$$

$$\Rightarrow \quad A = \mathcal{G}^{-1} \mathcal{H}$$

If we write $\mathcal{G}^{-1} = (g^{ij})$, then

$$\boxed{a_j^i = \sum_{k=1}^{2} g^{ik} h_{kj}}$$

Classical notation:

$$\begin{cases} E = \langle X_u, X_u \rangle = g_{11} \\ F = \langle X_u, X_v \rangle = g_{12} = g_{21} \\ G = \langle X_v, X_v \rangle = g_{22} \end{cases}$$

$$\begin{cases} l = \langle S(X_u), X_u \rangle = \langle X_{uu}, U \rangle = h_{11} \\ m = \langle S(X_u), X_v \rangle = \langle S(X_v), X_u \rangle = \langle X_{uv}, U \rangle = h_{12} = h_{21} \\ n = \langle S(X_v), X_v \rangle = \langle X_{vv}, U \rangle = h_{22} \end{cases}$$

i.e. $\mathcal{G} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ & $\mathcal{H} = \begin{pmatrix} l & m \\ m & n \end{pmatrix}$

Then $\mathcal{G}^{-1} = \dfrac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} = (g^{ij})$

i.e.

$$\begin{cases} g^{11} = \dfrac{G}{EG-F^2} \\[2ex] g^{21} = g^{12} = \dfrac{-F}{EG-F^2} \\[2ex] g^{22} = \dfrac{E}{EG-F^2} \end{cases}$$

Then

$$A = \begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix} = G^{-1} \mathcal{A} = \frac{1}{EG-F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} l & m \\ m & n \end{pmatrix}$$

$$= \frac{1}{EG-F^2} \begin{pmatrix} Gl - Fm & Gm - Fn \\ -Fl + Em & -Fm + En \end{pmatrix}$$

i.e.

$$\begin{cases} S(\mathbf{x}_u) = \left(\dfrac{Gl - Fm}{EG - F^2}\right)\mathbf{x}_u + \left(\dfrac{-Fl + Em}{EG - F^2}\right)\mathbf{x}_v \\[2ex] S(\mathbf{x}_v) = \left(\dfrac{Gm - Fn}{EG - F^2}\right)\mathbf{x}_u + \left(\dfrac{-Fm + En}{EG - F^2}\right)\mathbf{x}_v \end{cases}$$

(the a, b, c, d in the formula (2.3.1) of Oprea.)

<u>Note</u>: It is clear that $\mathcal{G}^{-1}$ exists $\Leftrightarrow$ $\det \mathcal{G} \neq 0$

i.e. $EG - F^2 \neq 0$ or $g_{11} g_{22} - g_{12}^2 \neq 0$

By definition, this is

$$\langle \mathbf{x}_u, \mathbf{x}_u \rangle \langle \mathbf{x}_v, \mathbf{x}_v \rangle - \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2 \neq 0$$

$$\Leftrightarrow \quad |\mathbf{x}_u|^2 |\mathbf{x}_v|^2 - \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2 \neq 0$$

$$\Leftrightarrow \quad |\mathbf{x}_u \times \mathbf{x}_v|^2 \neq 0$$

$\therefore$ The assumption that $\mathbf{x}$ is regular justifies our calculation.

Shape operator as differential of the Gauss map

Def (Differential of mapping between surfaces)

Let M, N be regular surfaces,

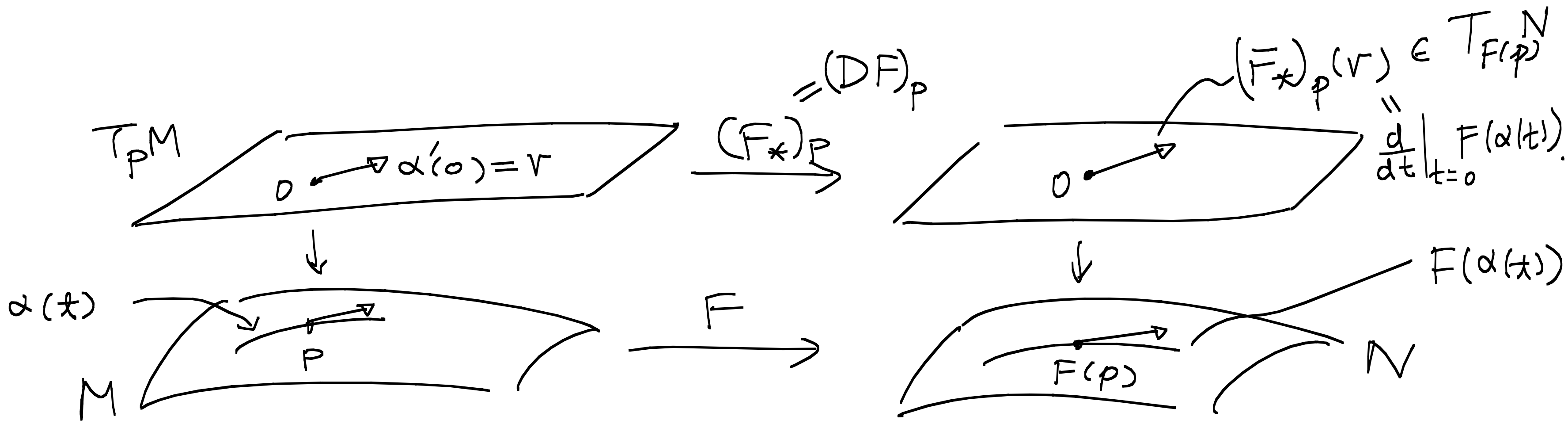
$F: M \to N$ a smooth mapping,

then we define the differential of F at a point $p \in M$

$$(DF)_p = (F_*)_p : T_pM \to T_{F(p)}N \quad \text{by}$$

(2 notations)

$$(F_*)_p(\alpha'(0)) = \frac{d}{dt}\Big|_{t=0} F(\alpha(t)), \quad \forall\, \alpha'(0) \in T_pM.$$



eg $F : \mathbb{R}^2 \to \mathbb{R}^2$

$(u,v) \mapsto (f(u,v), g(u,v))$

At $p = (u_0, v_0)$

$$(F_*)_p(\mathbf{X}_u) = \frac{d}{du}\Big|_{u=u_0} (f(u,v_0), g(u,v_0))$$

$$= \left(\frac{\partial f}{\partial u}(u_0,v_0), \frac{\partial g}{\partial u}(u_0,v_0)\right)$$

$$= f_u \mathbf{X}_u + g_u \mathbf{X}_v$$

where $\mathbf{X} : (u,v) \mapsto (u,v,0) \in \mathbb{R}^2 \subset \mathbb{R}^3$ is the standard coordinates patch on $\mathbb{R}^2$ when regarded as a surface in $\mathbb{R}^3$.

($\therefore$ $\mathbf{X}_u = (1,0)$ & $\mathbf{X}_v = (0,1)$)

Similarly $(F_*)_p(\mathbf{X}_v) = f_v \mathbf{X}_u + g_v \mathbf{X}_v$

$\Rightarrow$ matrix representation of $(F_*)_p$ in the standard basis $\{\overset{e_1}{\overset{\parallel}{\mathbf{X}_u}}, \overset{e_2}{\overset{\parallel}{\mathbf{X}_v}}\}$

is the Jacobian matrix $J(F) = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$. #

Consider the Gauss map $\mathcal{G} : M \to S^2$ defined by

$$\mathcal{G}(p) = U(p) \left(= \frac{\underline{X}_u \times \underline{X}_v}{|\underline{X}_u \times \underline{X}_v|}(p) \text{ wrt a patch } \underline{X}(u,v) \right)$$

$$\in S^2$$

Then $\forall\, v \in T_pM$ and a curve α on M with $\alpha(0) = p$ & $\alpha'(0) = v$,

we have

$$(\mathcal{G}_*)_p(v) = \frac{d}{dt}\Big|_{t=0} \mathcal{G}(\alpha(t))$$

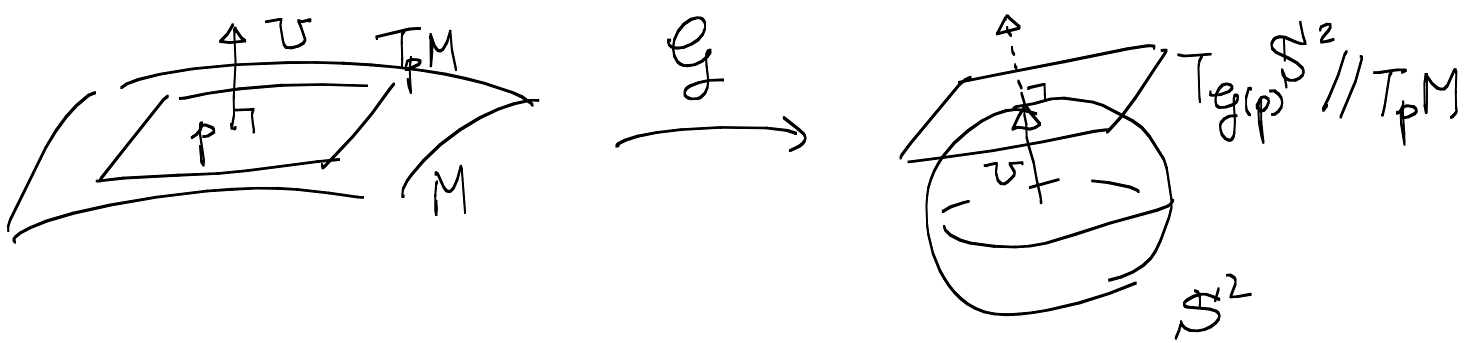
$$= \frac{d}{dt}\Big|_{t=0} U(\alpha(t))$$

$$= \nabla_{\alpha'(0)} U = -S_p(v).$$

$\therefore$ $\mathcal{G}_* = -S$ in the following sense:

Since $T_{\mathcal{G}(p)}S^2 \,/\!/\, T_pM$, $T_{\mathcal{G}(p)}S^2 \equiv T_pM$ as vector subspaces in $\mathbb{R}^3$.

$$\therefore\ (\mathcal{G}_*)_p :\ \begin{tikzcd} T_pM \arrow[r] \arrow[rr, bend right, "= -S_p"'] & T_{\mathcal{G}(p)}S^2 \cong & T_pM \end{tikzcd}$$



2.4 Normal curvature

Lemma (Lemma 2.4.1 of the text book)

$\forall$ curve α on M,

$$\langle \alpha'', U \rangle = \langle S(\alpha'), \alpha' \rangle,$$

where U = unit normal vector field (defining the orientation) &

S = corresponding shape operator.

Pf: By $\langle \alpha', U \rangle = 0$ $\quad (\alpha' \in TM, \ U \perp TM)$

$$\Rightarrow 0 = \alpha'[\langle \alpha', U \rangle]$$

$$= \frac{d}{dt} \langle \alpha'(t), U(\alpha(t)) \rangle$$

$$= \langle \alpha'', U \rangle + \langle \alpha', \frac{d}{dt} U(\alpha(t)) \rangle$$

$$= \langle \alpha'', U \rangle + \langle \alpha', \nabla_{\alpha'(t)} U \rangle$$

$$\Rightarrow \langle \alpha'', U \rangle = \langle \alpha', S(\alpha') \rangle$$

※

Remark: If α is parametrized by arc-length, ie. $|\alpha'|=1$, then $\alpha''=\kappa N$, where κ = curvature of α and N = principal normal of α.

Hence $\langle \alpha'', U \rangle = \kappa \langle N, U \rangle$ can be interpreted as the U-component of the curvature of α which is "normal" to the surface.

(Note: N may not normal to the surface!)

Def: $\forall$ unit vector $u \in T_pM$, the normal curvature of M in the u-direction is

$$\boxed{k(u) = \langle S_p(u), u \rangle}.$$

Remark : By the above lemma & remark,

if α is a curve s.t. $\alpha(0)=p$, $\alpha'(0)=\mathfrak{u}$,

then $$\boxed{k(\mathfrak{u}) = \kappa(0)\cos\theta}$$

where $\kappa(0)$ = curvature of α at p,

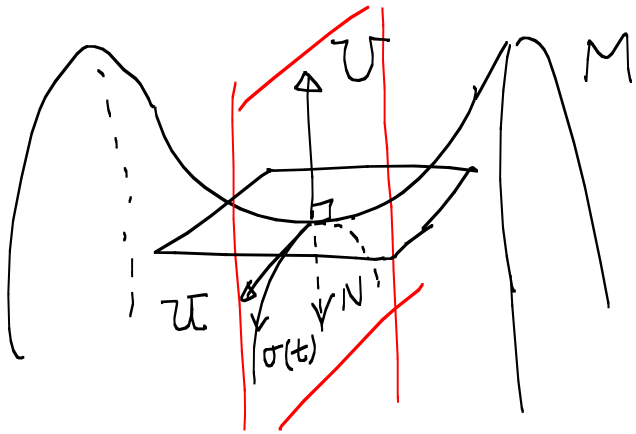
θ = angle between the principal normal N of α and the surface normal U.

Prop (Prop 2.4.3 of Oprea) $(|\mathfrak{u}|=1)$

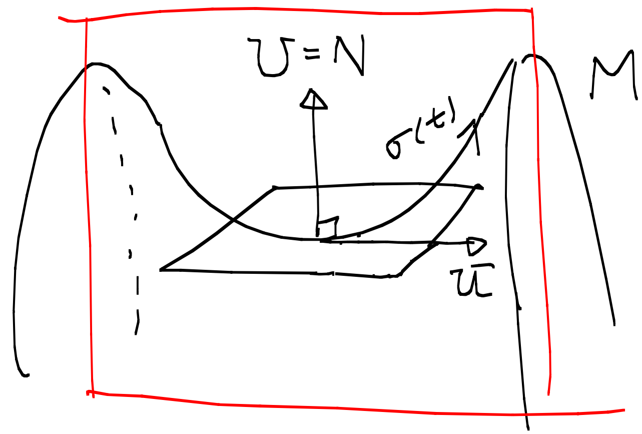
Let $\mathcal{P}$ = plane spanned by $U(p)$ & $\mathfrak{u}$, where $\mathfrak{u}\in T_pM$,

and let σ = unit speed curve formed by $\mathcal{P}\cap M$ with $\sigma(0)=p$.

Then $k(\mathfrak{u}) = \pm\,\kappa_\sigma(0)$ (where κ_σ = curvature of σ)



$$k(u) = -\kappa_\sigma(0) < 0$$



$$k(u) = +\kappa_\sigma(0) > 0$$

(Note: the sign of $k(u)$ depends on the choice of the surface normal U, i.e. the orientation of the surface.)

Pf: Note that the construction of σ

$\Rightarrow \quad \sigma'(0) \in \mathcal{P}$.

But $\sigma'(0) \in T_pM$, hence $\sigma'(0) = \pm u$ as they both have length 1

If necessary, we reverse the parametrization of σ

So that we have $\sigma'(0) = \mathcal{U}$.

(σ is parametrized by arc-length)

Now $\sigma \in \mathcal{P}$, i.e. σ is a plane curve,

$$\Rightarrow \quad N = \frac{\sigma''(0)}{\kappa_\sigma(0)} \in \mathcal{P} \quad \text{and} \quad N \perp \sigma'(0) = \mathcal{U}. \quad (\text{provided } \kappa_\sigma(0) > 0)$$

$$\Rightarrow \quad N = \pm \mathcal{U} \quad \Rightarrow \quad \theta = 0 \text{ or } \pi$$

$\uparrow$ angle between N & $\mathcal{U}$

$$\therefore \quad k(\mathcal{U}) = \kappa_\sigma(0) \cos\theta = \pm \kappa_\sigma(0).$$

(Then by continuity, this is true for the case $\kappa_\sigma(0) = 0$)

eg $M =$ saddle surface $z = x^2 - y^2$.

Then $\nu = \dfrac{(-2x, -2y, 1)}{\sqrt{1+4x^2+4y^2}}$ is the (surface) normal.

Let $p = (0,0,0) \in M$, then $\nu(p) = (0,0,1)$

$\Rightarrow$ $T_pM = \{ (t_1, t_2, 0) \in \mathbb{R}^3 \}$

Let $u = (1,0,0) \in S^1 \subset T_pM$.

Then the plane P determined by $\nu(p)$ & u is the xz-plane (i.e. $y \equiv 0$)

$\Rightarrow$ The intersection $P \cap M$ is $\sigma = \{ z = x^2 \}$.

Parametrize σ by $\sigma(t) = (t, 0, t^2)$.

Then $\sigma'(t) = (1, 0, 2t)$ has length $|\sigma'(t)| = \sqrt{1+4t^2}$

$\Rightarrow \quad s(t) = \int_0^t \sqrt{1+4t^2}\, dt$ is an arc-length function with inverse $t(s)$ s.t. $t(0)=0$.

Then $\quad \frac{d\sigma}{ds}(0) = \sigma'(t(0)) \frac{dt}{ds}(0) = (1, 0, 2t(0)) \cdot \frac{1}{\sqrt{1+4t^2(0)}}$

$$= (1,0,0) = u.$$

& $\quad \frac{d^2\sigma}{ds^2}(0) = \frac{d}{ds}\left[\sigma'(t)\frac{dt}{ds}\right]\Big|_{s=0}$

$$= \frac{d}{ds}\left(\frac{\sigma'(t)}{\sqrt{1+4t^2}}\right)\Big|_{s=0}$$

$$= \left[\frac{\sigma''(t)}{\sqrt{1+4t^2}} - \frac{\sigma'(t)\cdot 4t}{(1+4t^2)^{3/2}}\right]\cdot\frac{1}{\sqrt{1+4t^2}}\Big|_{s=0}$$

$$= \sigma''(0) = (0,0,2) = 2U(p)$$

$\therefore \quad K_\sigma(0) = 2 \quad \& \quad N(0) = U(p)$

Therefore $\quad k(u) = 2.$

Remark: $k : S^1 \subset T_pM \longrightarrow \mathbb{R}$ smooth

(normal curvature) compact

$\Rightarrow$ $\exists$ unit vectors v_1 & $v_2 \in T_pM$ s.t.

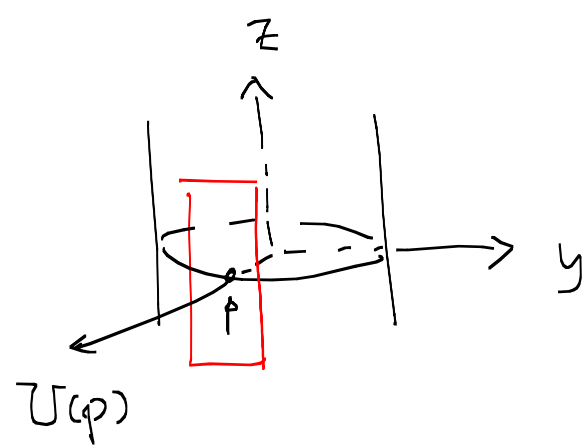
$$\begin{cases} k(v_1) = k_1 = \max\limits_{v \in S^1} k(v) \\ k(v_2) = k_2 = \min\limits_{v \in S^1} k(v). \end{cases}$$

- $\left\{ \begin{array}{l} v_1, v_2 \text{ are called } \underline{\text{principal vectors}} \text{ (or } \underline{\text{principal directions}}\text{) \&} \\ k_1, k_2 \text{ are called } \underline{\text{principal curvatures}}. \end{array} \right.$

<u>eg</u>: Let $M = \{ x^2 + y^2 = 1 \}$ be the cylinder.

Let $p = (1, 0, 0) \in M$.

Then $U(p) = (1, 0, 0)$



$\Rightarrow$ $T_pM = yz$-plane
(translated to $p = (1,0,0)$)

$\therefore$ $v \in T_pM \iff v = (0, \cos\varphi, \sin\varphi)$ for some φ.

Then the plane $\mathcal{P}$ determined by U & v passing thro. p
is

$$\langle (x,y,z) - p, U \times v \rangle = 0$$

$\uparrow$ normal to $\mathcal{P} = \text{span}\{U, v\}$

i.e $\langle (x,y,z) - (1,0,0), (1,0,0) \times (0, \cos\varphi, \sin\varphi) \rangle = 0$

$\iff$ $\langle (x-1, y, z), (0, -\sin\varphi, \cos\varphi) \rangle = 0$

$\iff$ $z\cos\varphi = y\sin\varphi$.

If $\cos\varphi \neq 0$, then $z = y\tan\varphi$

$\therefore\ \sigma = \mathcal{P} \cap M = \{ (\sqrt{1-y^2}, y, y\tan\varphi) \}$ near $p = (1,0,0)$

(this gives the $+$ sign in front of $\sqrt{1-y^2}$)

Therefore, σ can be parametrized by

$\sigma(t) = (\sqrt{1-t^2}, t, t\tan\varphi)$ for $|t| <$ small with

$\sigma(0) = (1,0,0) = p$

Then $\sigma'(t) = \left(\dfrac{-t}{\sqrt{1-t^2}}, 1, \tan\varphi \right)$

$$\sigma''(t) = \left(\frac{-1}{(1-t^2)^{3/2}}, 0, 0 \right)$$

$$\Rightarrow \underset{\text{(unit tangent of } \sigma)}{T(0)} = \frac{\sigma'(0)}{|\sigma'(0)|} = \frac{(0, 1, \tan\varphi)}{\sqrt{1+\tan^2\varphi}}$$

$$B(0) = \frac{\sigma'(0) \times \sigma''(0)}{|\sigma'(0) \times \sigma''(0)|}$$

(Binormal)

$$= \frac{(0, 1, \tan\varphi) \times (-1, 0, 0)}{|(0, 1, \tan\varphi) \times (-1, 0, 0)|}$$

$$= \frac{(0, -\tan\varphi, 1)}{\sqrt{1+\tan^2\varphi}}$$

$\Rightarrow$ principal normal $N(0) = B(0) \times T(0)$

$$= \frac{(0, -\tan\varphi, 1) \times (0, 1, \tan\varphi)}{1+\tan^2\varphi}$$

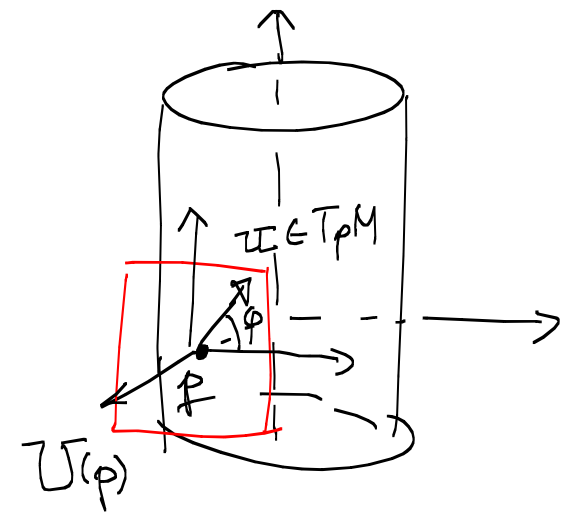
$$= (-1, 0, 0) = -\mho(p)$$

[We've proved that $\forall\ \varphi$ with $\cos\varphi \neq 0$, $N(0) = -\mho(p)$.

Then by continuity, $N(0) = -\mho(p)$ also for $\cos\varphi = 0$ (in certain sense).]

Finally, $$k_\sigma(0) = \frac{|\sigma'(0)\times\sigma''(0)|}{|\sigma'(0)|^3}$$

$$= \frac{\sqrt{1+\tan^2\varphi}}{\left(\sqrt{1+\tan^2\varphi}\right)^3} = \frac{1}{1+\tan^2\varphi} = \cos^2\varphi$$



$$\therefore \quad k(v) = k\big((0,\cos\varphi,\sin\varphi)\big) = -\cos^2\varphi$$

Consequently principal curvatures are

$$\begin{cases} k_1(p) = \max\limits_{v} k(v) = 0 \\ k_2(p) = \min\limits_{v} k(v) = -1 \end{cases}$$

with principal directions $\begin{cases} v_1 = \pm(0,0,1) \\ v_2 = \pm(0,1,0) \end{cases}$

Def: A point $p \in M$ is an umbilic point if $k_1(p) = k_2(p)$, where k_1, k_2 are principal curvatures.

Remark: By the definitions, p is umbilic $\Longrightarrow$

$$k(v) = k_1(p) = k_2(p) \quad \forall v \in T_pM, \ |v| = 1.$$

Thm (Thm 2.4.10 of Oprea)

(1) If $p \in M$ is umbilic, then $S_p(v) = k v \quad \forall v \in T_pM$, where $k = k_1 = k_2$ (depending on p).

(2) If $p \in M$ is not umbilic, then $\exists$ exactly 2 perpendicular unit eigenvectors of S_p with associated eigenvalues the principal curvatures at p.

Pf: Theory of linear algebra

Cor (Euler formula) (Cor 2.4.11 of Oprea)

Let $\vec{u}_1$ & $\vec{u}_2$ be 2 orthonormal eigenvectors of S_p s.t.

$$k_1 = k(\vec{u}_1) = \max k(\vec{u})$$

$$k_2 = k(\vec{u}_2) = \min k(\vec{u})$$

then for $\vec{u} = \cos\theta\, \vec{u}_1 + \sin\theta\, \vec{u}_2 \in S^1 \subset T_pM$, we have

$$k(\vec{u}) = \cos^2\theta\, k_1 + \sin^2\theta\, k_2$$

Remarks:
- The orthonormal eigenvectors are given directly from part (2) of the above theorem in the case of non-umbilic point. If p is umbilic, part (1) $\Rightarrow$ $S_p(\vec{u}) = k\vec{u}$ and we can choose any orthonormal basis $\{\vec{u}_1, \vec{u}_2\}$ of T_pM.
- Any $\vec{u} \in T_pM$ with $|\vec{u}| = 1$, we have $\vec{u} = \cos\theta\, \vec{u}_1 + \sin\theta\, \vec{u}_2$.

Pf : $k(\bar{v}) = \langle S_p(\bar{v}), \bar{v} \rangle$

$$= \langle S_p(\cos\theta\, \bar{v}_1 + \sin\theta\, \bar{v}_2),\ \cos\theta\, \bar{v}_1 + \sin\theta\, \bar{v}_2 \rangle$$

$$= \cos^2\theta\, k(\bar{v}_1) + \cos\theta \sin\theta \langle S_p(\bar{v}_1), \bar{v}_2 \rangle$$

$$+ \sin\theta\cos\theta \langle S_p(\bar{v}_2), \bar{v}_1 \rangle + \sin^2\theta\, k(\bar{v}_2)$$

By the above thm,

$$S_p(\bar{v}_1) = k_1 \bar{v}_1 \quad \& \quad S_p(\bar{v}_2) = k_2 \bar{v}_2 .$$

$$\Rightarrow k(\bar{v}) = \cos^2\theta \cdot k_1 + \cos\theta\sin\theta \left(\cancelto{0}{\langle k_1 \bar{v}_1, \bar{v}_2 \rangle} + \cancelto{0}{\langle k_2 \bar{v}_2, \bar{v}_1 \rangle} \right) + \sin^2\theta \cdot k_2$$

$$= \cos^2\theta\, k_1 + \sin^2\theta\, k_2 . \quad \times\!\!\!\times$$

Ch 3 Curvatures

3.1 Definition & Basis Facts

Def: Let S_p = shape operator of a surface M at a point p.

Then the Gauss (or Gaussian) curvature of M at $p \in M$

is defined by $\boxed{K(p) = \det(S_p)}$.

The mean curvature of M at $p \in M$ is defined by

$$\boxed{H(p) = \frac{1}{2}\,\mathrm{trace}(S_p)}.$$

Remarks • By Thm 2.4.10 of Oprea, the principal curvatures $k_1(p)$ & $k_2(p)$ are eigenvalues of S_p. Hence, wrt a basis of eigenvectors, S_p is represented by

$$\begin{bmatrix} k_1(p) & 0 \\ 0 & k_2(p) \end{bmatrix}.$$

Therefore,

$$\boxed{\begin{aligned} K(p) &= k_1(p)\,k_2(p) \\ H(p) &= \tfrac{1}{2}\left(k_1(p) + k_2(p)\right) \end{aligned}}$$

- Recall that k_1, k_2 depend on the choice of orientation, i.e. the unit normal U. If we take $-U$ as the surface normal (for instance, changing the order of (u,v) in a coordinate patch), then

$$k_{-U}(\vec{u}) = -k_U(\vec{u}) \quad \forall\, \vec{u} \in T_pM \text{ with } |\vec{u}| = 1.$$

$\therefore$ $k_i \rightsquigarrow -k_i$ as $U \to -U$.

Hence $\quad H(p) \rightsquigarrow -H(p) \quad$ but

$K(p)$ remains unchanged !

In fact $\mathrm{sign}(K(p))$ has a geometric meaning :

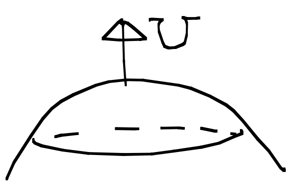
If $K>0$, then $k(v)$ have the same sign $\forall v \in T_pM$, $|v|=1$

(Ex)

$\Rightarrow$

$(k(v)>0, \forall v)$

or



$(k(v)<0, \forall v)$

If $K<0$, then k_1, k_2 have opposite sign

$\Rightarrow$

$(K=0\ ?)$

Facts (Ex)

(i) H = average of normal curvature

$$= \frac{1}{2\pi}\int_0^{2\pi} k(\theta)\,d\theta$$

where $k(\theta) = k(\cos\theta\, u_1 + \sin\theta\, u_2)$,

u_1, u_2 = principal directions.

(ii) $H = \dfrac{k(v_1) + k(v_2)}{2}$, $\forall$ orthonormal basis $\{v_1, v_2\}$ of T_pM.

(iii) $\begin{cases} k_1 = H + \sqrt{H^2 - K} \\ k_2 = H - \sqrt{H^2 - K} \end{cases}$

(note $H^2 - K \geq 0$)

Gauss map and curvature:

Recall that the differential $\mathcal{G}_*$ (a $D\mathcal{G}$) of the Gauss map $\mathcal{G}: M \to S^2$ is $-S$, i.e.

$$(\mathcal{G}_*)_p = -S_p$$

Then for a patch $\mathbf{x}(u,v)$,

$$(\mathcal{G}_*)_p(\mathbf{x}_u) \times (\mathcal{G}_*)_p(\mathbf{x}_v) = S_p(\mathbf{x}_u) \times S_p(\mathbf{x}_v)$$

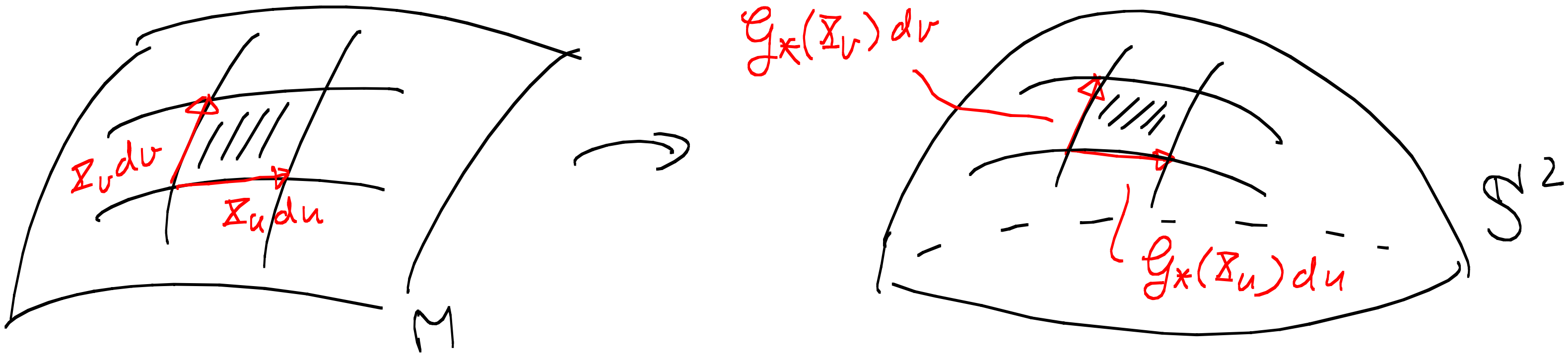
$$= (a_1^1 \mathbf{x}_u + a_1^2 \mathbf{x}_v) \times (a_2^1 \mathbf{x}_u + a_2^2 \mathbf{x}_v)$$

$$= (a_1^1 a_2^2 - a_1^2 a_2^1)\, \mathbf{x}_u \times \mathbf{x}_v$$

$$= K(p)\, \mathbf{x}_u \times \mathbf{x}_v$$

$$\Rightarrow \quad |K(p)| = \frac{|(\mathcal{G}_*)_p(\mathbf{x}_u) \times (\mathcal{G}_*)_p(\mathbf{x}_v)|}{|\mathbf{x}_u \times \mathbf{x}_v|}$$

$$= \lim_{\Omega \to p} \frac{|\text{Area } \mathcal{G}(\Omega)|}{|\text{Area}(\Omega)|} \qquad (\Omega = \text{region containing } p)$$



One may define "oriented area" and get the formula

$$\boxed{K(p) = \lim_{\Omega \to p} \frac{\text{Area } \mathcal{G}(\Omega)}{\text{Area}(\Omega)}}.$$

<u>Prop</u> (Prop. 3.1.8 of Oprea)

The (oriented) area of $\mathcal{G}(M)$ is equal to the <u>total Gauss curvature</u> of M, i.e. $\text{Area}(\mathcal{G}(M)) = \int_M K\,dA$, where $\mathcal{G}: M \to S^2$ is the Gauss map and

dA = area element of M. (in a patch $\mathbf{X}(u,v)$, $dA = |\mathbf{X}_u \times \mathbf{X}_v|\, du\, dv$)

Pf: We prove the situation that M is a patch $\mathbf{X}(u,v)$. Then

(oriented) area of $\mathcal{G}(M)$

normal to S^2 at $\mathcal{G}(p)$ = normal to M at p

$$\overset{\text{def}}{=} \int_M \langle \mathcal{G}_*(\mathbf{X}_u) \times \mathcal{G}_*(\mathbf{X}_v), U \rangle\, du\, dv$$

where $U = \dfrac{\mathbf{X}_u \times \mathbf{X}_v}{|\mathbf{X}_u \times \mathbf{X}_v|}$

$$= \int_M K \langle \mathbf{X}_u \times \mathbf{X}_v, U \rangle\, du\, dv$$

$$= \int_M K\, |\mathbf{X}_u \times \mathbf{X}_v|\, du\, dv$$

$$= \int_M K\, dA. \quad \#$$

3.2 Calculating Curvature

We recall the notations

Modern	Classical
$\mathbf{X}(u^1, u^2)$	$\mathbf{X}(u,v)$
$g_{ij} = \langle \mathbf{X}_i, \mathbf{X}_j \rangle$ (1st fundamental form)	$E = \langle \mathbf{X}_u, \mathbf{X}_u \rangle$, $F = \langle \mathbf{X}_u, \mathbf{X}_v \rangle = \langle \mathbf{X}_v, \mathbf{X}_u \rangle$, $G = \langle \mathbf{X}_v, \mathbf{X}_v \rangle$
$h_{ij} = \langle \mathbf{X}_{ij}, U \rangle = \langle S(\mathbf{X}_i), \mathbf{X}_j \rangle$ (2nd fundamental form)	$l = \langle \mathbf{X}_{uu}, U \rangle = \langle S(\mathbf{X}_u), \mathbf{X}_u \rangle$, $m = \langle \mathbf{X}_{uv}, U \rangle = \langle S(\mathbf{X}_u), \mathbf{X}_v \rangle = \langle S(\mathbf{X}_v), \mathbf{X}_u \rangle = \langle \mathbf{X}_{vu}, U \rangle$, $n = \langle \mathbf{X}_{vv}, U \rangle = \langle S(\mathbf{X}_v), \mathbf{X}_v \rangle$
$\mathbf{X}_i = \frac{\partial \mathbf{X}}{\partial u^i}$, $\mathbf{X}_{ij} = \frac{\partial^2 \mathbf{X}}{\partial u^i \partial u^j}$ etc	$\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}$, $\mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v}$, $\mathbf{X}_{uv} = \frac{\partial^2 \mathbf{X}}{\partial u \partial v}$, etc

Matrix representation of S (wrt $\{X_1, X_2\}$)

<u>modern</u>

$$\begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix} = (g^{ij})(h_{ij})$$

$$= (g_{ij})^{-1}(h_{ij})$$

$$= \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

$$= \frac{1}{(g_{11}g_{22} - g_{12}^2)} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

$$\boxed{a^i_j = \sum_{k=1}^{2} g^{ik} h_{kj}}$$

<u>classical</u>

$$\begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix}$$

$$= \frac{1}{EG - F^2} \begin{pmatrix} G\ell - Fm & Gm - Fn \\ -F\ell + Em & -Fm + En \end{pmatrix}$$

Therefore, formulae for K & H are

Modern	Classical
$K = \det\begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix}$ $= \det(g_{ij})^{-1}\det(h_{ij})$ $\boxed{K = \dfrac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}}$	$K = \det\begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix}$ $\boxed{K = \dfrac{ln - m^2}{EG - F^2}}$
$H = \frac{1}{2}\operatorname{tr}\begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix}$ $\boxed{H = \dfrac{g_{22}h_{11} - 2g_{12}h_{12} + g_{11}h_{22}}{2(g_{11}g_{22} - g_{12}^2)}}$ (Ex!)	$H = \frac{1}{2}\operatorname{tr}\begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix}$ $\boxed{H = \dfrac{lG - 2mF + nE}{2(EG - F^2)}}$

eg <u>Enneper's Surface</u> :

$$\mathbb{X}(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right)$$

$$\Rightarrow \begin{cases} \mathbb{X}_1 = \mathbb{X}_u = (1-u^2+v^2,\ 2uv,\ 2u) \\ \mathbb{X}_2 = \mathbb{X}_v = (2uv,\ 1-v^2+u^2,\ -2v) \end{cases}$$

$$\Rightarrow \begin{cases} \mathbb{X}_{11} = \mathbb{X}_{uu} = (-2u,\ 2v,\ 2) \\ \mathbb{X}_{12} = \mathbb{X}_{uv} = (2v,\ 2u,\ 0) \\ \mathbb{X}_{22} = \mathbb{X}_{vv} = (2u,\ -2v,\ -2) \end{cases}$$

$$\Rightarrow \begin{cases} g_{11} = E = \langle \mathbb{X}_1, \mathbb{X}_1 \rangle = (1-u^2+v^2)^2 + (2uv)^2 + (2u)^2 \\ \qquad\qquad = (1+u^2+v^2)^2 \\ g_{12} = F = \langle \mathbb{X}_1, \mathbb{X}_2 \rangle = (1-u^2+v^2)\cdot 2uv + 2uv(1-v^2+u^2) + 2u(-2v) \\ \qquad\qquad = 0 \\ g_{22} = G = \langle \mathbb{X}_2, \mathbb{X}_2 \rangle = (1+u^2+v^2)^2 \end{cases}$$

$$\mathbf{X}_1 \times \mathbf{X}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1-u^2+v^2 & 2uv & 2u \\ 2uv & 1-v^2+u^2 & -2v \end{vmatrix}$$

$$= \left(-2u(1+u^2+v^2),\ 2v(1+u^2+v^2),\ 1-(u^2+v^2)^2\right)$$

$$\Rightarrow \quad |\mathbf{X}_1 \times \mathbf{X}_2|^2 = (1+u^2+v^2)^4$$

(or using $|\mathbf{X}_1 \times \mathbf{X}_2|^2 = g_{11}g_{22} - g_{12}^2$)

$$\Rightarrow \quad U = \frac{\mathbf{X}_1 \times \mathbf{X}_2}{|\mathbf{X}_1 \times \mathbf{X}_2|} = \left(\frac{-2u}{1+u^2+v^2},\ \frac{2v}{1+u^2+v^2},\ \frac{1-(u^2+v^2)^2}{(1+u^2+v^2)^2} \right)$$

Therefore

$$h_{11} = l = \langle \mathbf{X}_{11}, U \rangle$$

$$= (-2u)\left(\frac{-2u}{1+u^2+v^2}\right) + (2v)\left(\frac{2v}{1+u^2+v^2}\right) + 2 \cdot \frac{1-(u^2+v^2)^2}{(1+u^2+v^2)^2}$$

$$= 2$$

$$h_{12} = m = \langle \mathbb{X}_{12}, U \rangle$$

$$= (2v)\left(\frac{-2u}{1+u^2+v^2}\right) + (2u)\left(\frac{2v}{1+u^2+v^2}\right) + 0 \cdot \frac{1-(u^2+v^2)}{(1+u^2+v^2)^2}$$

$$= 0$$

$$h_{22} = n = (2u)\left(\frac{-2u}{1+u^2+v^2}\right) + (-2v)\left(\frac{2v}{1+u^2+v^2}\right) + (-2)\,\frac{1-(u^2+v^2)^2}{(1+u^2+v^2)^2}$$

$$= -2$$

In summary

$$\begin{cases} (g_{ij}) = \begin{pmatrix} (1+u^2+v^2)^2 & 0 \\ 0 & (1+u^2+v^2)^2 \end{pmatrix} \\ \\ (h_{ij}) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \end{cases}$$

$\therefore$ The Gauss curvature $K = \dfrac{\det(h_{ij})}{\det(g_{ij})} = \dfrac{-4}{(1+u^2+v^2)^4}$, $\#$

the mean curvature $H = \frac{1}{2} \operatorname{tr}\left[(g_{ij})^{-1} (h_{ij}) \right]$

$$= \frac{1}{2} \operatorname{tr} \begin{pmatrix} \frac{2}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{-2}{(1+u^2+v^2)^2} \end{pmatrix}$$

$$= 0$$

($\therefore$ Enneper's surface is a <u>minimal surface</u> $(H \equiv 0)$.)

eg: Torus

$$X(u,v) = ((R+r\cos u)\cos v,\ (R+r\cos u)\sin v,\ r\sin u)$$

$$(R > r > 0)$$

$$\Rightarrow \begin{cases} X_1 = (-r\sin u\cos v,\ -r\sin u\sin v,\ r\cos u) \\ X_2 = (-(R+r\cos u)\sin v,\ (R+r\cos u)\cos v,\ 0) \end{cases}$$

$$\Rightarrow \begin{cases} X_{11} = (-r\cos u\cos v,\ -r\cos u\sin v,\ -r\sin u) \\ X_{12} = (r\sin u\sin v,\ -r\sin u\cos v,\ 0) \\ X_{22} = (-(R+r\cos u)\cos v,\ -(R+r\cos u)\sin v,\ 0) \end{cases}$$

&

$$\begin{cases} g_{11} = r^2\sin^2 u\cos^2 v + r^2\sin^2 u\sin^2 v + r^2\cos^2 u = r^2 \\ g_{12} = r\sin u\cos v\,(R+r\cos u)\sin v - r\sin u\sin v\,(R+r\cos u)\cos v = 0 \\ g_{22} = (R+r\cos u)^2\sin^2 u + (R+r\cos u)^2\cos^2 v = (R+r\cos u)^2 \end{cases}$$

$$\therefore\quad |X_1 \times X_2|^2 = g_{11}g_{22} - g_{12}^2 = r^2(R+r\cos u)^2$$

& $h_{11} = \langle X_{11}, U \rangle = \langle X_{11}, \dfrac{X_1 \times X_2}{|X_1 \times X_2|} \rangle = \dfrac{1}{|X_1 \times X_2|} \langle X_{11}, X_1 \times X_2 \rangle$

$$= \frac{1}{|X_1 \times X_2|} \det \begin{pmatrix} - X_{11} - \\ - X_1 - \\ - X_2 - \end{pmatrix}$$

$$= \frac{1}{r(R + r\cos u)} \det \begin{vmatrix} -r\cos u \cos v & -r\cos u \sin v & -r\sin u \\ -r\sin u \cos v & -r\sin u \sin v & r\cos u \\ -(R+r\cos u)\sin v & (R+r\cos u)\cos v & 0 \end{vmatrix}$$

(Ex.) $\quad = r$

$$h_{12} = \langle X_{12}, U \rangle = \frac{1}{r(R + r\cos u)} \det \begin{vmatrix} r\sin u \sin v & -r\sin u \cos v & 0 \\ -r\sin u \cos v & -r\sin u \sin v & r\cos u \\ -(R+r\cos u)\sin v & (R+r\cos u)\cos v & 0 \end{vmatrix}$$

$= 0$

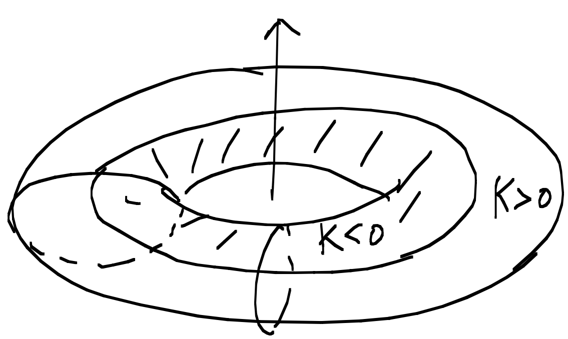
$$h_{22} = \langle X_{22}, U \rangle = \frac{1}{r(R + r\cos u)} \det \begin{vmatrix} -(R+r\cos u)\cos v & -(R+r\cos u)\sin v & 0 \\ -r\sin u \cos v & -r\sin u \sin v & r\cos u \\ -(R+r\cos u)\sin v & (R+r\cos u)\cos v & 0 \end{vmatrix}$$

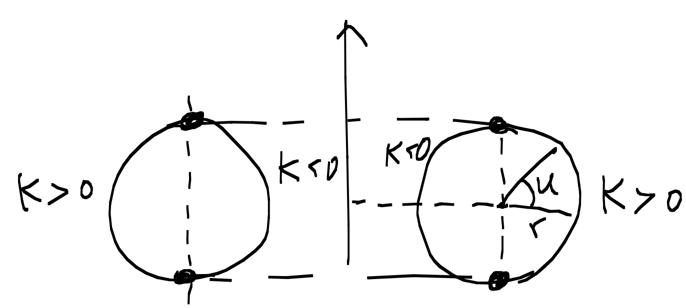
$= (R + r\cos u)\cos u$

In summary $(g_{ij}) = \begin{pmatrix} r^2 & 0 \\ 0 & (R+r\cos u)^2 \end{pmatrix}$

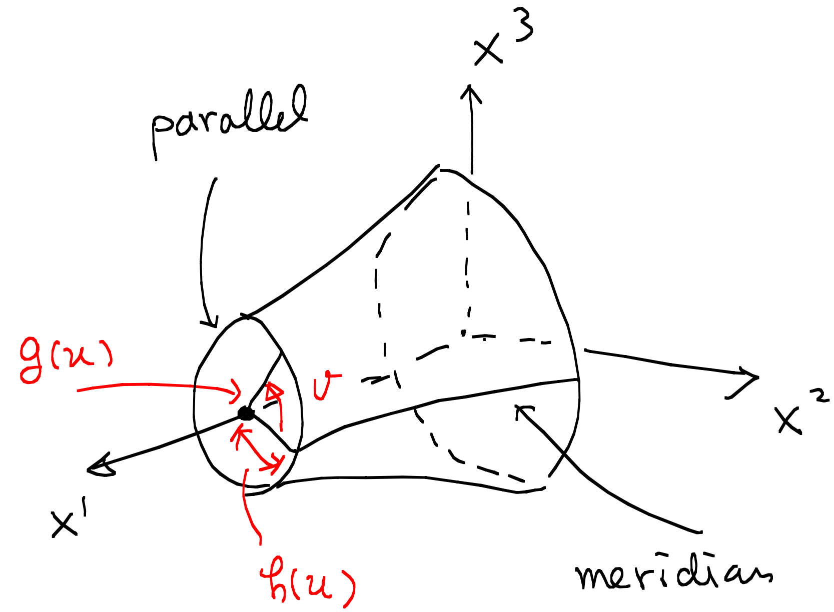
$$(h_{ij}) = \begin{pmatrix} r & 0 \\ 0 & (R+r\cos u)\cos u \end{pmatrix}$$

$$\Rightarrow \begin{cases} K = \dfrac{r\,(R+r\cos u)\cos u}{r^2\,(R+r\cos u)^2} = \dfrac{\cos u}{r(R+r\cos u)} \\ H = \dfrac{1}{2}\left(\dfrac{r}{r^2} + \dfrac{(R+r\cos u)\cos u}{(R+r\cos u)^2}\right) = \dfrac{1}{2}\left(\dfrac{1}{r} + \dfrac{\cos u}{R+r\cos u}\right). \end{cases}$$





3.3 Surfaces of Revolution



$$X(u,v) = (g(u), h(u)\cos v, h(u)\sin v)$$

$g(u)$ = coordinate on the rotational axis,

$h(u)$ = radius of the parallel

v = angle

Then
$$\begin{cases} X_1 = (g', h'\cos v, h'\sin v) \\ X_2 = (0, -h\sin v, h\cos v) \end{cases}$$

$$\begin{cases} X_{11} = (g'', h''\cos v, h''\sin v) \\ X_{12} = (0, -h'\sin v, h'\cos v) \\ X_{22} = (0, -h\cos v, -h\sin v) \end{cases}$$

$$\Rightarrow \begin{cases} g_{11} = \langle \mathbf{X}_1, \mathbf{X}_1 \rangle = g'^2 + h'^2 \\ g_{12} = \langle \mathbf{X}_1, \mathbf{X}_2 \rangle = 0 \\ g_{22} = \langle \mathbf{X}_2, \mathbf{X}_2 \rangle = h^2 \end{cases}$$

$$\therefore \quad (g_{ij}) = \begin{pmatrix} g'^2 + h'^2 & 0 \\ 0 & h^2 \end{pmatrix} \Rightarrow (g_{ij})^{-1} = (g^{ij}) = \begin{pmatrix} \frac{1}{g'^2+h'^2} & 0 \\ 0 & \frac{1}{h^2} \end{pmatrix}$$

$$\& \quad |\mathbf{X}_1 \times \mathbf{X}_2|^2 = \det(g_{ij}) = h^2(g'^2 + h'^2)$$

$$h_{11} = \left\langle \mathbf{X}_{11}, \frac{\mathbf{X}_1 \times \mathbf{X}_2}{|\mathbf{X}_1 \times \mathbf{X}_2|} \right\rangle = \frac{1}{h\sqrt{g'^2+h'^2}} \det \begin{pmatrix} g'' & h'' \cos v & h'' \sin v \\ g' & h' \cos v & h' \sin v \\ 0 & -h \sin v & h \cos v \end{pmatrix}$$

$$= \frac{1}{h\sqrt{g'^2+h'^2}} \left[g''(hh' \cos^2 v + hh' \sin^2 v) - g'(hh'' \cos^2 v + hh'' \sin^2 v) \right]$$

$$= \frac{g''h' - g'h''}{\sqrt{g'^2+h'^2}}$$

$$h_{12} = \left\langle X_{12}, \frac{X_1 \times X_2}{|X_1 \times X_2|} \right\rangle = \frac{1}{h\sqrt{g'^2+h'^2}} \det \begin{pmatrix} 0 & -h'\sin v & h'\cos v \\ g' & h'\cos v & h'\sin v \\ 0 & -h\sin v & h\cos v \end{pmatrix}$$

$$= 0$$

$$h_{22} = \left\langle X_{22}, \frac{X_1 \times X_2}{|X_1 \times X_2|} \right\rangle = \frac{1}{h\sqrt{g'^2+h'^2}} \det \begin{pmatrix} 0 & -h\cos v & -h\sin v \\ g' & h'\cos v & h'\sin v \\ 0 & -h\sin v & h\cos v \end{pmatrix}$$

$$= \frac{1}{h\sqrt{g'^2+h'^2}} (-1)g' \left[(-h\cos v)(h\cos v) - (-h\sin v)(-h\sin v) \right]$$

$$= \frac{h g'}{\sqrt{g'^2+h'^2}}$$

$$\therefore \quad (h_{ij}) = \begin{pmatrix} \dfrac{g''h' - g'h''}{\sqrt{g'^2+h'^2}} & 0 \\ 0 & \dfrac{h g'}{\sqrt{g'^2+h'^2}} \end{pmatrix}$$

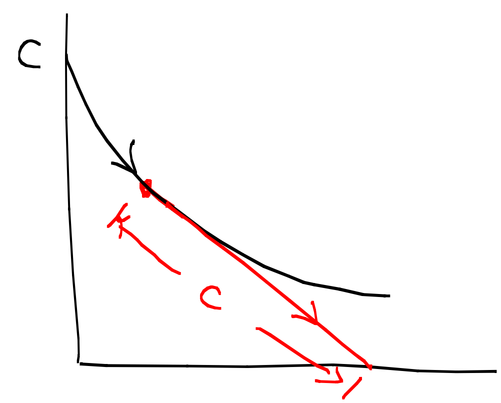
Hence

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{h g'(g''h' - g'h'')}{(g'^2 + h'^2)} \Big/ h^2(g'^2 + h'^2)$$

$$\Rightarrow \quad \boxed{K = \frac{g'(g''h' - g'h'')}{h\,(g'^2 + h'^2)^2}}$$

eg: <u>Pseudosphere</u> (surface of revolution obtained by revolving the <u>tractrix</u>)

<u>Tractrix</u>: a curve which begins at $(0,c)$ and traces a path so that its tangent line at any point reaches the (positive) x-axis after running a distance exactly equal to c.



Differential equation for the tractrix:

Parametrize the tractrix by $(u, h(u))$, then the tangent line of the tractrix at $(u, h(u))$ is

$$(u, h(u)) + t(1, h'(u)) \quad , \quad t \in \mathbb{R}$$

$\Rightarrow$ x-intersection of the tangent line has y-coordinate

$$h(u) + t\, h'(u) = 0$$

ie. $t = -\frac{h}{h'}$

$\therefore$ the x-intersection point is $\left(u - \frac{h}{h'}, 0\right)$ & $h' < 0$.

Hence by definition of tractrix,

$$C = \left| (u, h) - \left(u - \frac{h}{h'}, 0\right) \right| = \left| \left(\frac{h}{h'}, h\right) \right|$$

$$= h \sqrt{\frac{1}{h'^2} + 1} \, .$$

$$\Rightarrow \quad h'^2 = \frac{h^2}{c^2 - h^2}$$

$$\Rightarrow \quad \boxed{h' = -\frac{h}{\sqrt{c^2 - h^2}}} \qquad (h' \neq 0)$$

$\therefore$ The Pseudosphere is parametrized by

$$\gamma(u,v) = (u,\ h(u)\cos v,\ h(u)\sin v) \qquad (\text{note: } g(u) = u)$$

with h satisfying $h' = -\dfrac{h}{\sqrt{c^2 - h^2}}$ for some $c > 0$.

Therefore $$K = \frac{g'(g''h' - g'h'')}{h(g'^2 + h'^2)^2}$$

$$= \frac{-h''}{h(1 + h'^2)^2}. \qquad (g' = 1,\ g'' = 0)$$

Now $\quad h' = -\dfrac{h}{\sqrt{c^2 - h^2}} \quad \Rightarrow \quad 1 + h'^2 = 1 + \dfrac{h^2}{c^2 - h^2} = \dfrac{c^2}{c^2 - h^2}$

and $\quad 2h'h'' = \dfrac{2c^2 h h'}{(c^2 - h^2)^2}$

$$\therefore \quad h'' = \frac{c^2 h}{(c^2 - h^2)^2}.$$

Hence

$$K = \frac{-\dfrac{c^2 h}{(c^2-h^2)^2}}{h \cdot \left(\dfrac{c^2}{c^2-h^2}\right)^2} = -\frac{1}{c^2} \quad \text{(constant negative curvature)}$$

<u>eg</u>: Surface of revolution with <u>constant Gauss curvature</u>.

Let M be parametrized by

$$\mathbf{x}(u,v) = (g(u),\, h(u)\cos v,\, h(u)\sin v)$$

with $\quad g'(u)^2 + h'(u)^2 = 1 \qquad$ (u = arc-length of the curve)

Then $\quad g'g'' + h'h'' = 0$

$$\Rightarrow \quad K = \frac{g'(g''h' - g'h'')}{h\,(g'^2 + h'^2)^2} = \frac{(g'g'')h' - g'^2 h''}{h}$$

$$= \frac{(-h'h'')h' - (1 - h'^2)h''}{h}$$

$$= -\frac{h''}{h}$$

$\therefore \quad h$ must satisfies $\quad \boxed{h'' + Kh = 0}$.

If $K =$ constant, then

$$h'' + K h = 0$$

$$\Rightarrow \quad h = \begin{cases} A\cos(\sqrt{K}u) + B\sin(\sqrt{K}u), & K > 0 \\ Au + B, & K = 0 \\ A\cosh(\sqrt{-K}u) + B\sinh(\sqrt{-K}u), & K < 0 \end{cases}$$

Then $g(u)$ is given by $\quad g'(u) = \pm\sqrt{1 - h'(u)^2}$

$$\Rightarrow \quad g(u) = \pm \int_{u_0}^{u} \sqrt{1 - h'(u)^2}\, du.$$

Explicit eg : If $K \equiv 0$, then $h = Au + B$

$$\Rightarrow \quad h' = A \Rightarrow A^2 < 1 \ \& \ g' = \pm\sqrt{1 - A^2}.$$

$$\Rightarrow \quad g = \pm\sqrt{1 - A^2}\,(u - u_0)$$

Therefore $\quad (g(u), h(u)) = \left(\pm\sqrt{1-A^2}(u - u_0),\ Au + B\right)$

$$= \left(\pm\sqrt{1-A^2},\ A\right)u + \left(\mp\sqrt{1-A^2}\,u_0,\ B\right)$$

is a straight line.

$\therefore$ The surface of revolution is either a cone or a cylinder

Supplementary material : Curvatures of graphs.

Monge patch : $\mathbf{X}(u,v) = (u, v, f(u,v))$

$$\Rightarrow \begin{cases} \mathbf{X}_1 = (1, 0, f_1) \\ \mathbf{X}_2 = (0, 1, f_2) \end{cases}$$

$$\begin{cases} \mathbf{X}_{11} = (0, 0, f_{11}) \\ \mathbf{X}_{12} = (0, 0, f_{12}) \\ \mathbf{X}_{22} = (0, 0, f_{22}) \end{cases}$$

$$\Rightarrow \begin{cases} g_{11} = \langle \mathbf{X}_1, \mathbf{X}_1 \rangle = 1 + f_1^2 \\ g_{12} = \langle \mathbf{X}_1, \mathbf{X}_2 \rangle = f_1 f_2 \quad (\neq 0 \text{ in general}) \\ g_{22} = \langle \mathbf{X}_2, \mathbf{X}_2 \rangle = 1 + f_2^2 \end{cases}$$

& $$|\mathbf{X}_1 \times \mathbf{X}_2|^2 = \det(g_{ij}) = (1+f_1^2)(1+f_2^2) - f_1^2 f_2^2 = 1 + f_1^2 + f_2^2$$
$$= 1 + |\nabla f|^2 .$$

$$
\begin{cases}
h_{11} = \left\langle X_{11}, \dfrac{X_1 \times X_2}{|X_1 \times X_2|} \right\rangle = \dfrac{1}{\sqrt{1+|\nabla f|^2}} \det \begin{pmatrix} 0 & 0 & f_{11} \\ 1 & 0 & f_1 \\ 0 & 1 & f_2 \end{pmatrix} \\
\quad = \dfrac{f_{11}}{\sqrt{1+|\nabla f|^2}} \\
h_{12} = \left\langle X_{12}, \dfrac{X_1 \times X_2}{|X_1 \times X_2|} \right\rangle = \dfrac{1}{\sqrt{1+|\nabla f|^2}} \det \begin{pmatrix} 0 & 0 & f_{12} \\ 1 & 0 & f_1 \\ 0 & 1 & f_2 \end{pmatrix} \\
\quad = \dfrac{f_{12}}{\sqrt{1+|\nabla f|^2}} \\
h_{22} = \left\langle X_{22}, \dfrac{X_1 \times X_2}{|X_1 \times X_2|} \right\rangle = \dfrac{1}{\sqrt{1+|\nabla f|^2}} \det \begin{pmatrix} 0 & 0 & f_{22} \\ 1 & 0 & f_1 \\ 0 & 1 & f_2 \end{pmatrix} \\
\quad = \dfrac{f_{22}}{\sqrt{1+|\nabla f|^2}}
\end{cases}
$$

Then
$$\begin{cases} K = \dfrac{\det(h_{ij})}{\det(g_{ij})} = \dfrac{f_{11}f_{22} - f_{12}^2}{(1+|\nabla f|^2)^2} \\[2ex] H = \dfrac{1}{2}\,\dfrac{g_{22}h_{11} - 2g_{12}h_{12} + g_{11}h_{22}}{\det(g_{ij})} \end{cases}$$

$$= \frac{1}{2}\,\frac{(1+f_2^2)f_{11} - 2f_1f_2f_{12} + (1+f_1^2)f_{22}}{(1+|\nabla f|^2)^{3/2}}$$

It is also nice to know the surface normal

$$U = \frac{X_1 \times X_2}{|X_1 \times X_2|} = \frac{(-f_1, -f_2, 1)}{\sqrt{1+|\nabla f|^2}}.$$

3.4 A formula for Gauss Curvature (Theorema Egregium)

(More than the ref book by Oprea)

Def For a regular surface M with coordinate patch $\mathbf{X}(u^1, u^2)$, the functions

$$g_{ij} = \langle \mathbf{X}_i, \mathbf{X}_j \rangle$$

are called metric coefficients of M, or the coefficients of the 1^{st} fundamental form.

Thm (Gauss' Theorema Egregium)

The Gauss curvature depends only on the metric coefficients (and their derivatives.)

(in particular, independent on the "2^{nd} fundamental form" $h_{ij} = \langle \mathbf{X}_{ij}, U \rangle$)

<u>Christoffel symbols</u>:

For the patch $X(u^1,u^2)$, $\{X_1, X_2, U\}$, where $U = \dfrac{X_1 \times X_2}{|X_1 \times X_2|}$,

forms a basis of $\mathbb{R}^3$ (at a point). Therefore.

$$\boxed{X_{ij} = \sum_{k=1}^{2} \Gamma^k_{ij} X_k + h_{ij} U} \quad \forall\, i,j = 1,2$$

for some coefficients $\{\Gamma^k_{ij}\}_{i,j,k=1,2}$, where $h_{ij} = \langle X_{ij}, U \rangle$

as $\langle X_k, U \rangle = 0$ for $k = 1,2$.

These coefficients called <u>Christoffel symbols</u> of the metric (g_{ij}) of M.

<u>Prop</u> Γ^k_{ij} is symmetric in $\{i,j\}$.

Pf: Using definition, $X_{ij} = \sum_{\ell} \Gamma^{\ell}_{ij} X_{\ell} + h_{ij} U$. Then $\langle U, X_k \rangle = 0$,

$$\Rightarrow \quad \langle X_{ij}, X_k \rangle = \sum_\ell \Gamma^\ell_{ij} \langle X_\ell, X_k \rangle = \sum_\ell \Gamma^\ell_{ij} g_{\ell k}$$

$$\Rightarrow \quad \sum_k \langle X_{ij}, X_k \rangle g^{ks} = \sum_k \left(\sum_\ell \Gamma^\ell_{ij} g_{\ell k} g^{ks} \right)$$

$$= \sum_\ell \Gamma^\ell_{ij} \left(\sum_k g_{\ell k} g^{ks} \right)$$

$$= \sum_\ell \Gamma^\ell_{ij} \delta^s_\ell \qquad \left(\text{since } (g^{ks}) = (g_{ij})^{-1} \right)$$

$$= \Gamma^s_{ij}$$

$$\therefore \quad \Gamma^k_{ij} = \sum_\ell g^{k\ell} \langle X_{ij}, X_\ell \rangle$$

$$= \sum_\ell g^{k\ell} \langle X_{ji}, X_\ell \rangle = \Gamma^k_{ji} \qquad (\text{since } X_{ij} = X_{ji}) \quad \#$$

<u>Prop</u>

$$\boxed{\Gamma^k_{ij} = \frac{1}{2} \sum_{\ell=1}^{2} g^{k\ell} \left(\frac{\partial g_{\ell j}}{\partial u^i} + \frac{\partial g_{i\ell}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\ell} \right)} \qquad \forall\, i, j, k = 1, 2$$

where $(g^{k\ell}) = (g_{ij})^{-1}$.

Pf: By def., $g_{ij} = \langle X_i, X_j \rangle$

$$\Rightarrow \quad \frac{\partial g_{ij}}{\partial u^k} = \langle X_{ik}, X_j \rangle + \langle X_i, X_{jk} \rangle$$

$$= \langle \sum_l \Gamma_{ik}^l X_l + h_{ik} U, X_j \rangle + \langle X_i, \sum_l \Gamma_{jk}^l X_l + h_{jk} U \rangle$$

$$\therefore \quad \frac{\partial g_{ij}}{\partial u^k} = \sum_l \Gamma_{ik}^l g_{lj} + \sum_l \Gamma_{jk}^l g_{il} \quad \text{——} (1)$$

Similarly $$\frac{\partial g_{jk}}{\partial u^i} = \sum_l \Gamma_{ji}^l g_{lk} + \sum_l \Gamma_{ki}^l g_{jl} \quad \text{——} (2)$$

$$\frac{\partial g_{ki}}{\partial u^j} = \sum_l \Gamma_{kj}^l g_{li} + \sum_l \Gamma_{ij}^l g_{kl} \quad \text{——} (3)$$

Hence using symmetry of Γ_{ij}^k & g_{ij} (in $\{i, j\}$),

we have, by $(2) + (3) - (1)$,

$$\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} = 2 \sum_l \Gamma_{ij}^l g_{lk}$$

$$\Rightarrow \quad \frac{1}{2}\sum_k g^{ks}\left(\frac{\partial g_{jk}}{\partial u^i}+\frac{\partial g_{ki}}{\partial u^j}-\frac{\partial g_{ij}}{\partial u^k}\right) = \sum_k\left(\sum_\ell \Gamma_{ij}^\ell g_{\ell k}\right)g^{ks}$$

$$= \sum_\ell \Gamma_{ij}^\ell \left(\sum_k g_{\ell k} g^{ks}\right)$$

$$= \sum_\ell \Gamma_{ij}^\ell \delta_\ell^s = \Gamma_{ij}^s$$

$$\Rightarrow \quad \Gamma_{ij}^k = \frac{1}{2}\sum_\ell g^{k\ell}\left(\frac{\partial g_{\ell j}}{\partial u^i}+\frac{\partial g_{i\ell}}{\partial u^j}-\frac{\partial g_{ij}}{\partial u^\ell}\right). \quad \#$$

<u>Cor</u>: $\{\Gamma_{ij}^k\}$ depends only on (g_{ij}).

Now consider the 3rd derivatives of X:

$$X_{ijk} = \frac{\partial}{\partial u^k}\left(\sum_\ell \Gamma_{ij}^\ell X_\ell + h_{ij} U\right)$$

$$= \sum_\ell \frac{\partial \Gamma_{ij}^\ell}{\partial u^k} X_\ell + \sum_\ell \Gamma_{ij}^\ell X_{\ell k} + \frac{\partial h_{ij}}{\partial u^k} U + h_{ij}\frac{\partial U}{\partial u^k}.$$

Note: $\frac{\partial U}{\partial u^k}$ is in fact given by the Shape operator. However, we'll re-do the calculation as follows:

$$\langle U, U \rangle = 1 \Rightarrow \langle \frac{\partial U}{\partial u^k}, U \rangle = 0$$

$$\Rightarrow \quad \frac{\partial U}{\partial u^k} = \sum_{l=1}^{2} b_k^l X_l \qquad \forall\, k = 1, 2$$

for some coefficients $\{b_k^l\}$.

$$\Rightarrow \quad \langle \frac{\partial U}{\partial u^k}, X_i \rangle = \sum_l b_k^l \langle X_l, X_i \rangle = \sum_l b_k^l g_{li}$$

Since $\langle U, X_i \rangle = 0$, $\langle \frac{\partial U}{\partial u^k}, X_i \rangle + \langle U, X_{ik} \rangle = 0$

$$\therefore \quad -h_{ik} = \sum_l b_k^l g_{li}$$

$$\Rightarrow \quad \sum_i (-h_{ik} g^{is}) = \sum_i \left(\sum_l b_k^l g_{li}\right) g^{is} = \sum_l b_k^l \sum_i g_{li} g^{is}$$

$$= \sum_l b_k^l \delta_l^s = b_k^s$$

$$\therefore \quad b_k^{\,l} = -\sum_s g^{ls} h_{sk} \qquad \left(= -a_k^{\,l} \leftarrow \text{entries of the matrix representation of the shape operator} \right)$$

Substitute back into the formula for $\mathbf{X}_{ijk}$,

$$\mathbf{X}_{ijk} = \sum_l \frac{\partial \Gamma_{ij}^{l}}{\partial u^k} \mathbf{X}_l + \sum_s \Gamma_{ij}^{s} \left(\sum_l \Gamma_{sk}^{l} \mathbf{X}_l + h_{sk} \mathbf{U} \right) + \frac{\partial h_{ij}}{\partial u^k} \mathbf{U}$$

$$+ h_{ij} \left(\sum_l \left(-\sum_s g^{ls} h_{sk} \right) \mathbf{X}_l \right)$$

$$= \sum_l \left(\frac{\partial \Gamma_{ij}^{l}}{\partial u^k} + \sum_s \Gamma_{ij}^{s} \Gamma_{sk}^{l} - \sum_s h_{ij} h_{ks} g^{sl} \right) \mathbf{X}_l$$

$$+ \left(\frac{\partial h_{ij}}{\partial u^k} + \sum_l \Gamma_{ij}^{l} h_{lk} \right) \mathbf{U}$$

(Note: We've interchanged the dummy indexes l & s in the 2nd term and the symmetry of g_{ij} & h_{ij}, etc. in order to obtain the above formula.)

Since X is smooth, we have $X_{ijk} = \dfrac{\partial^3 X}{\partial u^i \partial u^j \partial u^k} = X_{ikj}$ and hence

$$0 = \sum_\ell \left[\frac{\partial \Gamma^\ell_{ij}}{\partial u^k} - \frac{\partial \Gamma^\ell_{ik}}{\partial u^j} + \sum_s \left(\Gamma^s_{ij} \Gamma^\ell_{sk} - \Gamma^s_{ik} \Gamma^\ell_{sj} \right) - \sum_s \left(h_{ij} h_{ks} - h_{ik} h_{js} \right) g^{s\ell} \right] X_\ell$$

$$+ \left[\left(\frac{\partial h_{ij}}{\partial u^k} - \frac{\partial h_{ik}}{\partial u^j} \right) + \sum_\ell \left(\Gamma^\ell_{ij} h_{\ell k} - \Gamma^\ell_{ik} h_{\ell j} \right) \right] U$$

Since $\{X_1, X_2, U\}$ is a basis, we have $\forall\, i, j, k = 1, 2$,

(Gauss Equations) $$\sum_s \left(h_{ij} h_{ks} - h_{ik} h_{js} \right) g^{s\ell} = \frac{\partial \Gamma^\ell_{ij}}{\partial u^k} - \frac{\partial \Gamma^\ell_{ik}}{\partial u^j} + \sum_s \left(\Gamma^s_{ij} \Gamma^\ell_{sk} - \Gamma^s_{ik} \Gamma^\ell_{sj} \right)$$

(Codazzi-Mainardi Equations) $$\frac{\partial h_{ij}}{\partial u^k} + \sum_\ell \Gamma^\ell_{ij} h_{\ell k} = \frac{\partial h_{ik}}{\partial u^j} + \sum_\ell \Gamma^\ell_{ik} h_{\ell j}$$

These are the <u>equations of compatibility</u> (section §4.3 of do Carmo)

The Gauss equations $\Rightarrow$

$$\sum_{l}\sum_{s}(h_{ij}h_{ks} - h_{ik}h_{js})g^{sl}g_{lr} = \sum_{l} g_{lr}\left[\frac{\partial \Gamma^{l}_{ij}}{\partial u^k} - \frac{\partial \Gamma^{l}_{ik}}{\partial u^j} + \sum_{s}(\Gamma^{s}_{ij}\Gamma^{l}_{sk} - \Gamma^{s}_{ik}\Gamma^{l}_{sj})\right]$$

$$\Rightarrow \quad h_{ij}h_{kr} - h_{ik}h_{jr} = \sum_{l} g_{lr}\left[\frac{\partial \Gamma^{l}_{ij}}{\partial u^k} - \frac{\partial \Gamma^{l}_{ik}}{\partial u^j} + \sum_{s}(\Gamma^{s}_{ij}\Gamma^{l}_{sk} - \Gamma^{s}_{ik}\Gamma^{l}_{sj})\right]$$

Putting $i = j = 1$, $k = r = 2$, we have

$$h_{11}h_{22} - h_{12}h_{12} = \sum_{l} g_{l2}\left[\frac{\partial \Gamma^{l}_{11}}{\partial u^2} - \frac{\partial \Gamma^{l}_{12}}{\partial u^1} + \sum_{s}(\Gamma^{s}_{11}\Gamma^{l}_{s2} - \Gamma^{s}_{12}\Gamma^{l}_{s1})\right]$$

$$\Rightarrow \boxed{K = \frac{1}{\det(g_{ij})}\sum_{l=1}^{2} g_{l2}\left[\frac{\partial \Gamma^{l}_{11}}{\partial u^2} - \frac{\partial \Gamma^{l}_{12}}{\partial u^1} + \sum_{s=1}^{2}(\Gamma^{s}_{11}\Gamma^{l}_{s2} - \Gamma^{s}_{12}\Gamma^{l}_{s1})\right]}$$

Since the RHS is completely determined by the metric coefficients and their derivatives, we've proved the <u>Theorema Egregium</u>.

A formula for the Gauss curvature in the special case : $g_{12}=0$.

If $g_{12}=0$, then $(g^{ij}) = \begin{pmatrix} \frac{1}{g_{11}} & 0 \\ 0 & \frac{1}{g_{22}} \end{pmatrix}$

$$\Rightarrow \left\{ \begin{aligned} \Gamma^1_{11} &= \frac{1}{2} g^{11}\left(\frac{\partial g_{11}}{\partial u^1} + \frac{\partial g_{11}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^1}\right) + \frac{1}{2}\cancelto{0}{g^{12}}(\cdots) \\ &= \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial u^1} \\ \Gamma^2_{11} &= \frac{1}{2}\cancelto{0}{g^{21}}(\cdots) + \frac{1}{2} g^{22}\left(\cancelto{0}{\frac{\partial g_{21}}{\partial u^1}} + \cancelto{0}{\frac{\partial g_{12}}{\partial u^1}} - \frac{\partial g_{11}}{\partial u^2}\right) \end{aligned} \right.$$

$$= -\frac{1}{2g_{22}}\frac{\partial g_{11}}{\partial u^2}$$

$$\left\{ \begin{aligned} \Gamma^1_{21} = \Gamma^1_{12} &= \frac{1}{2} g^{11}\left(\cancelto{0}{\frac{\partial g_{12}}{\partial u^1}} + \frac{\partial g_{11}}{\partial u^2} - \cancelto{0}{\frac{\partial g_{12}}{\partial u^1}}\right) + \frac{1}{2}\cancelto{0}{g^{12}}(\cdots) = \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial u^2} \\ \Gamma^2_{21} = \Gamma^2_{12} &= \frac{1}{2}\cancelto{0}{g^{21}}(\cdots) + \frac{1}{2} g^{22}\left(\frac{\partial g_{22}}{\partial u^1} + \cancelto{0}{\frac{\partial g_{12}}{\partial u^2}} - \cancelto{0}{\frac{\partial g_{12}}{\partial u^2}}\right) = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial u^1} \end{aligned} \right.$$

$$\begin{cases} \Gamma^1_{22} = \frac{1}{2} g^{11}\left(\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^1}\right) + 0 = \frac{-1}{2g_{11}} \frac{\partial g_{22}}{\partial u^1} \\ \Gamma^2_{22} = 0 + \frac{1}{2} g^{22}\left(\frac{\partial g_{22}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^2}\right) = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial u^2} \end{cases}$$

In classical notation

$$\begin{cases} \Gamma^u_{uu} = \frac{E_u}{2E}, & \Gamma^v_{uu} = -\frac{E_v}{2G} \\ \Gamma^u_{uv} = \Gamma^u_{vu} = \frac{E_v}{2E}, & \Gamma^v_{uv} = \Gamma^v_{vu} = \frac{G_u}{2G} \\ \Gamma^u_{vv} = -\frac{G_u}{2E}, & \Gamma^v_{vv} = \frac{G_v}{2G} \end{cases}$$

Now by the formula for the Gauss curvature

$$K = \frac{1}{\det(g_{ij})} \sum_{l=1}^{2} g_{l2} \left[\frac{\partial \Gamma^l_{11}}{\partial u^2} - \frac{\partial \Gamma^l_{12}}{\partial u^1} + \sum_{s=1}^{2} \left(\Gamma^s_{11} \Gamma^l_{s2} - \Gamma^s_{12} \Gamma^l_{s1} \right) \right]$$

$$= \frac{1}{g_{11} g_{22}} \cdot g_{22} \left[\frac{\partial \Gamma^2_{11}}{\partial u^2} - \frac{\partial \Gamma^2_{12}}{\partial u^1} + \sum_{s=1}^{2} \left(\Gamma^s_{11} \Gamma^2_{s2} - \Gamma^s_{12} \Gamma^2_{s1} \right) \right]$$

$$= \frac{1}{g_{11}} \left[\frac{\partial \Gamma^2_{11}}{\partial u^2} - \frac{\partial \Gamma^2_{12}}{\partial u^1} + \left(\Gamma^1_{11} \Gamma^2_{12} - \Gamma^1_{12} \Gamma^2_{11} \right) + \left(\Gamma^2_{11} \Gamma^2_{22} - \Gamma^2_{12} \Gamma^2_{21} \right) \right]$$

The terms without derivatives give (in classical notations)

$$\frac{1}{g_{11}} \left(\Gamma^1_{11} \Gamma^2_{12} - \Gamma^1_{12} \Gamma^2_{11} + \Gamma^2_{11} \Gamma^2_{22} - \Gamma^2_{12} \Gamma^2_{21} \right)$$

$$= \frac{1}{E} \left[\frac{E_u}{2E} \cdot \frac{G_u}{2G} - \frac{E_v}{2E} \cdot \left(-\frac{E_v}{2G} \right) + \left(-\frac{E_v}{2G} \right) \cdot \frac{G_v}{2G} - \left(\frac{G_u}{2G} \right)^2 \right]$$

$$= \frac{E_u G_u}{4E^2 G} + \frac{(E_v)^2}{4E^2 G} - \frac{E_v G_v}{4EG^2} - \frac{G_u^2}{4EG^2}.$$

The terms with derivatives give (in classical notations)

$$\frac{1}{g_{11}} \left[\frac{\partial \Gamma^2_{11}}{\partial u^2} - \frac{\partial \Gamma^2_{12}}{\partial u^1} \right] = \frac{1}{E} \left[\frac{\partial}{\partial v} \left(-\frac{E_v}{2G} \right) - \frac{\partial}{\partial u} \left(\frac{G_u}{2G} \right) \right]$$

$$= \frac{-1}{2E}\left[\frac{\partial}{\partial v}\left(\sqrt{\frac{E}{G}}\cdot\frac{E_v}{\sqrt{EG}}\right)+\frac{\partial}{\partial u}\left(\sqrt{\frac{E}{G}}\cdot\frac{G_u}{\sqrt{EG}}\right)\right]$$

$$= -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)+\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right]$$

$$-\frac{1}{2E}\frac{\partial}{\partial v}\left(\sqrt{\frac{E}{G}}\right)\frac{E_v}{\sqrt{EG}}-\frac{1}{2E}\frac{\partial}{\partial u}\left(\sqrt{\frac{E}{G}}\right)\frac{G_u}{\sqrt{EG}}$$

$$= -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)+\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right]$$

$$-\frac{1}{2E}\cdot\frac{1}{2\sqrt{\frac{E}{G}}}\left(\frac{GE_v-EG_v}{G^2}\right)\frac{E_v}{\sqrt{EG}}$$

$$-\frac{1}{2E}\cdot\frac{1}{2\sqrt{\frac{E}{G}}}\left(\frac{GE_u-EG_u}{G^2}\right)\frac{G_u}{\sqrt{EG}}$$

$$= -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)+\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right]$$

$$- \frac{E_v^2}{4E^2G} + \frac{E_v G_v}{4EG^2} - \frac{E_u G_u}{4E^2G} + \frac{G_u^2}{4EG^2}$$

Hence $\boxed{K = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right], \text{ provided } F=0}$

(This is the Theorem 3.4.1 of Oprea)

In modern notation, if $\boxed{g_{12}=0,}$ then

$$\boxed{K = -\frac{1}{2\sqrt{g_{11}g_{22}}}\left[\frac{\partial}{\partial u^2}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{11}}{\partial u^2}\right) + \frac{\partial}{\partial u^1}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{22}}{\partial u^1}\right)\right]}$$

For the general case that $g_{12} \neq 0$ (i.e. $F \neq 0$), we have the following formula in Classical notation (ex. 3.4.7 of Oprea)

$$K = \frac{1}{(EG-F^2)^2}\left\{ \begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix} \right\}$$

In modern notation, that is

$$K = \frac{1}{\det(g_{ij})^2}\left\{ \begin{vmatrix} -\frac{1}{2}\frac{\partial^2 g_{11}}{(\partial u^1)^2} + \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{(\partial u^2)^2} & \frac{1}{2}\frac{\partial g_{11}}{\partial u^1} & \frac{\partial g_{12}}{\partial u^1} - \frac{1}{2}\frac{\partial g_{11}}{\partial u^2} \\ \frac{\partial g_{12}}{\partial u^2} - \frac{1}{2}\frac{\partial g_{22}}{\partial u^1} & g_{11} & g_{12} \\ \frac{1}{2}\frac{\partial g_{22}}{\partial u^2} & g_{21} & g_{22} \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}\frac{\partial g_{11}}{\partial u^2} & \frac{1}{2}\frac{\partial g_{22}}{\partial u^1} \\ \frac{1}{2}\frac{\partial g_{11}}{\partial u^2} & g_{11} & g_{12} \\ \frac{1}{2}\frac{\partial g_{22}}{\partial u^1} & g_{12} & g_{22} \end{vmatrix} \right\}$$

Pf: $h_{ij} = \langle X_{ij}, U \rangle = \dfrac{\langle X_{ij}, X_1 \times X_2 \rangle}{|X_1 \times X_2|} = \dfrac{\begin{vmatrix} X_{ij} \\ X_1 \\ X_2 \end{vmatrix}}{|X_1 \times X_2|}$

$$= \frac{\begin{vmatrix} X_{ij} & X_1 & X_2 \end{vmatrix}}{|X_1 \times X_2|}$$

$\Rightarrow$

$$h_{11} h_{22} = \frac{\begin{vmatrix} X_{11} \\ X_1 \\ X_2 \end{vmatrix} \begin{vmatrix} X_{22} & X_1 & X_2 \end{vmatrix}}{|X_1 \times X_2|^2}$$

$$= \frac{1}{\det(g_{ij})} \begin{vmatrix} \langle X_{11}, X_{22} \rangle & \langle X_{11}, X_1 \rangle & \langle X_{11}, X_2 \rangle \\ \langle X_1, X_{22} \rangle & \langle X_1, X_1 \rangle & \langle X_1, X_2 \rangle \\ \langle X_2, X_{22} \rangle & \langle X_2, X_1 \rangle & \langle X_2, X_2 \rangle \end{vmatrix}$$

$$\therefore \det(g_{ij})\, h_{11} h_{22} = \left| \begin{array}{c|cc} \langle X_{11}, X_{22} \rangle & \langle X_{11}, X_1 \rangle & \langle X_{11}, X_2 \rangle \\ \hline \langle X_1, X_{22} \rangle & \multicolumn{2}{c}{\multirow{2}{*}{g_{ij}}} \\ \langle X_2, X_{22} \rangle & & \end{array} \right|$$

Similarly

$$\det(g_{ij})\, h_{12}^2 = \left| \begin{array}{c|cc} \langle X_{12}, X_{12} \rangle & \langle X_{12}, X_1 \rangle & \langle X_{12}, X_2 \rangle \\ \hline \langle X_1, X_{12} \rangle & \multicolumn{2}{c}{\multirow{2}{*}{g_{ij}}} \\ \langle X_2, X_{12} \rangle & & \end{array} \right|$$

$$\det^2(g_{ij})\, K = \left| \begin{array}{c|cc} \langle X_{11}, X_{22} \rangle & \langle X_{11}, X_1 \rangle & \langle X_{11}, X_2 \rangle \\ \hline \langle X_1, X_{22} \rangle & \multicolumn{2}{c}{\multirow{2}{*}{g_{ij}}} \\ \langle X_2, X_{22} \rangle & & \end{array} \right| - \left| \begin{array}{c|cc} \langle X_{12}, X_{12} \rangle & \langle X_{12}, X_1 \rangle & \langle X_{12}, X_2 \rangle \\ \hline \langle X_1, X_{12} \rangle & \multicolumn{2}{c}{\multirow{2}{*}{g_{ij}}} \\ \langle X_2, X_{12} \rangle & & \end{array} \right|$$

$$= \left| \begin{array}{c|cc} \langle X_{11}, X_{22}\rangle - \langle X_{12}, X_{12}\rangle & \langle X_{11}, X_1\rangle & \langle X_{11}, X_2\rangle \\ \hline \langle X_1, X_{22}\rangle & & \\ \langle X_2, X_{22}\rangle & \multicolumn{2}{c}{g_{ij}} \end{array} \right| - \left| \begin{array}{c|cc} 0 & \langle X_{12}, X_1\rangle & \langle X_{12}, X_2\rangle \\ \hline \langle X_1, X_{12}\rangle & & \\ \langle X_2, X_{12}\rangle & \multicolumn{2}{c}{g_{ij}} \end{array} \right|$$

Now
$$\begin{cases} \langle X_{12}, X_1\rangle = \frac{1}{2}\langle X_1, X_1\rangle_2 = \frac{1}{2}\frac{\partial g_{11}}{\partial u^2} \\ \langle X_{12}, X_2\rangle = \frac{1}{2}\langle X_2, X_2\rangle_1 = \frac{1}{2}\frac{\partial g_{22}}{\partial u^1} \end{cases}$$

$$\begin{cases} \langle X_{11}, X_1\rangle = \frac{1}{2}\langle X_1, X_1\rangle_1 = \frac{1}{2}\frac{\partial g_{11}}{\partial u^1} \\ \langle X_{11}, X_2\rangle = \langle X_1, X_2\rangle_1 - \langle X_1, X_{21}\rangle = \frac{\partial g_{12}}{\partial u^1} - \frac{1}{2}\langle X_1, X_1\rangle_2 \\ \qquad = \frac{\partial g_{12}}{\partial u^1} - \frac{1}{2}\frac{\partial g_{11}}{\partial u^2} \end{cases}$$

$$\begin{cases} \langle X_1, X_{22}\rangle = \langle X_1, X_2\rangle_2 - \langle X_{12}, X_2\rangle = \frac{\partial g_{12}}{\partial u^2} - \frac{1}{2}\frac{\partial g_{22}}{\partial u^1} \\ \langle X_2, X_{22}\rangle = \frac{1}{2}\langle X_2, X_2\rangle_2 = \frac{1}{2}\frac{\partial g_{22}}{\partial u^2} \end{cases}$$

$$\langle X_{11}, X_{22}\rangle - \langle X_{12}, X_{12}\rangle$$

$$= \langle X_1, X_{22}\rangle_1 - \langle X_1, X_{221}\rangle$$

$$- \langle X_1, X_{12}\rangle_2 + \langle X_1, X_{122}\rangle$$

$= 0$ by $X_{221} = X_{122}$

$$= \frac{\partial}{\partial u^1}\left(\frac{\partial g_{12}}{\partial u^2} - \frac{1}{2}\frac{\partial g_{22}}{\partial u^1}\right) - \frac{\partial}{\partial u^2}\left(\frac{1}{2}\frac{\partial g_{11}}{\partial u^2}\right)$$

$$= \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{(\partial u^1)^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{(\partial u^2)^2}$$

This completes the proof. $\#$

3.5 Some Effects of Curvature

Thm (Thm 3.5.2 of Oprea)

A surface M consisting entirely of umbilic point is contained in either a plane or a sphere. ($H^2 = K$)

Pf: Umbilic at all points

$\Rightarrow$ $k_1(p) = k_2(p) = k(p)$ for some function $k(p)$ on M.

$$\Rightarrow \begin{cases} \nabla_{X_1} U = -S(X_1) = -k X_1 \\ \nabla_{X_2} U = -S(X_2) = -k X_2 \end{cases} \quad \forall p \in M,$$

where $U = \dfrac{X_1 \times X_2}{|X_1 \times X_2|}$ & S = shape operator.

$$\Rightarrow \begin{cases} \nabla_{X_2} \nabla_{X_1} U = -(\nabla_{X_2} k) X_1 - k \nabla_{X_2} X_1 \\ \nabla_{X_1} \nabla_{X_2} U = -(\nabla_{X_1} k) X_2 - k \nabla_{X_1} X_2 , \end{cases}$$

It is clear that
$$\begin{cases} \nabla_{X_2} X_1 = X_{12} = \nabla_{X_1} X_2 \ \& \\ \nabla_{X_2} \nabla_{X_1} U = \nabla_{X_1} \nabla_{X_2} U, \end{cases}$$

$$\therefore \quad (\nabla_{X_2} k) X_1 = (\nabla_{X_1} k) X_2 .$$

Then $\{X_1, X_2\}$ is a basis $\Rightarrow$ $\nabla_{X_1} k = \nabla_{X_2} k = 0$

$\Rightarrow$ k = constant

<u>Case 1</u>. If $k=0$, then $S \equiv 0 \Rightarrow M$ contained in a plane.

<u>Case 2</u>. If $k \neq 0$, then we consider

$$X + \frac{1}{k} U .$$

Then
$$\frac{\partial}{\partial u^i}\left(X + \frac{1}{k} U\right) = X_i + \frac{1}{k} \nabla_{X_i} U$$
$$= X_i - \frac{1}{k} S(X_i)$$
$$= X_i - \frac{1}{k} \cdot k X_i = 0$$

$\therefore \quad \mathbf{x} + \frac{1}{k} U = $ constant point p_0

$\Rightarrow \quad |\mathbf{x} - p_0| = |-\frac{1}{k} U| = \frac{1}{|k|}$ constant.

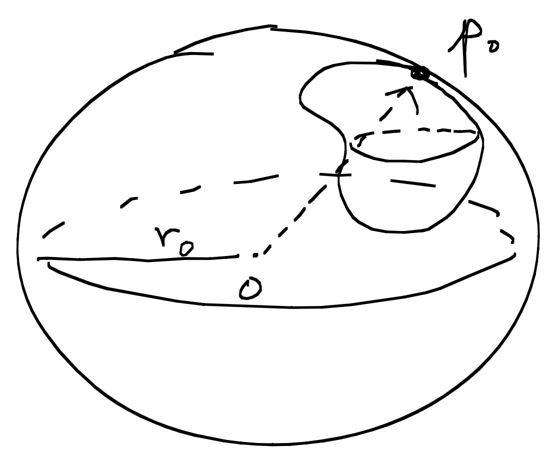
$\therefore$ M contained in a sphere. ※

Thm (Thm 3.5.3 of Oprea)

On every compact surface $M \subseteq \mathbb{R}^3$, there is some point $p \in M$ with $K(p) > 0$.

Pf: M compact

$\Rightarrow \exists$ a sphere of radius r_0, $S^2_{r_0}$, s.t. M contained in the interior of $S^2_{r_0}$ and touch $S^2_{r_0}$ at some point p_0.



i.e. $\exists\ p_0 \in M$ such that $|p_0|^2 = \max\limits_{p\in M} |p|^2 = r_0^2$.

Let $v \in T_{p_0}M$ & $|v| = 1$.

And let α = curve on M s.t.

$$\begin{cases} \alpha(0) = p_0 \text{ and} \\ \alpha'(0) = v. \end{cases}$$

Then $|\alpha(t)|^2 \leq |p_0|^2 = |\alpha(0)|^2$

$$\Rightarrow \quad \begin{cases} \left.\dfrac{d}{dt}\right|_{t=0} |\alpha(t)|^2 = 0 \quad \& \\ \left.\dfrac{d^2}{dt^2}\right|_{t=0} |\alpha(t)|^2 \leq 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} \langle \alpha(0), \alpha'(0) \rangle = 0 \\ \langle \alpha(0), \alpha''(0) \rangle + \langle \alpha'(0), \alpha'(0) \rangle \leq 0 \end{cases}$$

i.e.
$$\begin{cases} \langle p_0, v \rangle = 0 \\ \langle p_0, \alpha''(0) \rangle \leq -|v|^2 = -1. \end{cases}$$

Now $\quad k(v) = \langle S_{p_0}(v), v \rangle$

$$= \langle S_{p_0}(\alpha'(0)), \alpha'(0) \rangle$$

$$= \langle \alpha''(0), U(p_0) \rangle$$

where U = surface normal

Note that at p_0, M tangent to $S^2_{r_0}$

$$\Rightarrow \quad T_{p_0}M \mathbin{/\!/} T_{p_0}S^2_{r_0}$$

$$\therefore \quad U(p_0) = \frac{p_0}{|p_0|} = \frac{p_0}{r_0}$$

(as $K(p_0)$ is independent of the choice of U, we may assume the orientation of M is the one s.t. $U(p_0)$ is pointing outward)

$$\therefore \quad k(v) = \langle \alpha''(0), \frac{p_0}{r_0} \rangle \leq \frac{-1}{r_0}, \quad \forall\ v \in T_{p_0}M \text{ with } |v| = 1$$

In particular, $k_1(p_0), k_2(p_0) \leq -\frac{1}{r_0}$

and hence $K(p_0) = k_1(p_0)\, k_2(p_0) \geq \frac{1}{r_0^2} > 0$. $\#$

Cor (cor 3.5.4 of Oprea)

There is no compact surface in $\mathbb{R}^3$ with $K \leq 0$. In particular, no minimal surface embedded in $\mathbb{R}^3$ is compact.

(Note: minimal surface $\overset{\text{def}}{=}$ surface with $H \equiv 0$.
"embedded" $=$ regular surface in $\mathbb{R}^3$ without self intersection.)

Pf: 1^{st} statement follows immediately from the theorem.

For the 2^{nd} statement, minimal $\Leftrightarrow H \equiv 0$

$\therefore k_1 + k_2 = 0 \Rightarrow K = -k_1^2 \leq 0$. $\#$

Def: A curve $\alpha: I \to M$ is a *line of curvature* if $\alpha'(t)$ is a *principal direction*, i.e., an eigenvector of the Shape Operator $S_{\alpha(t)}$, at $\alpha(t)$ $\forall\, t \in I$.

eg. Surface of revolution parametrized by

$$\mathbf{X}(u,v) = (g(u),\ h(u)\cos v,\ h(u)\sin v).$$

Then the *meridians* (denoted μ) $= \mathbf{X}(u, v_0)$, $v_0 = \text{const}$,

the *parallels* (denoted π) $= \mathbf{X}(u_0, v)$, $u_0 = \text{const}$,

are line of curvatures.

In fact, we've shown that

$$(g_{ij}) = \begin{pmatrix} g'^2 + h'^2 & 0 \\ 0 & h^2 \end{pmatrix}$$

$$(h_{ij}) = \begin{pmatrix} \dfrac{g''h' - h''g'}{\sqrt{g'^2+h'^2}} & 0 \\ 0 & \dfrac{h\,g'}{\sqrt{g'^2+h'^2}} \end{pmatrix}$$

$\Rightarrow$ the matrix representing the Shape operator is

$$A = (g_{ij})^{-1}(h_{ij}) = \begin{pmatrix} \dfrac{g''h' - h''g'}{(g'^2+h'^2)^{3/2}} & 0 \\ 0 & \dfrac{g'}{h\sqrt{g'^2+h'^2}} \end{pmatrix}$$

$$= \begin{pmatrix} k_\mu & 0 \\ 0 & k_\pi \end{pmatrix}$$

where $k_\mu = \dfrac{g''h' - h''g'}{(g'^2+h'^2)^{3/2}}$ & $k_\pi = \dfrac{g'}{h\sqrt{g'^2+h'^2}}$ are

the principal curvatures (in the meridian & parallel directions respectively).

$$\Rightarrow \quad S(X_u) = k_\mu X_u \quad \& \quad S(X_v) = k_\pi X_v$$

$\therefore$ Both meridians and parallels are line of curvature of a surface of revolution. $\#$

The fact that a surface of revolution is parametrized in such a way that the u- & v- parameter curves (i.e. coordinate curves) are lines of curvature is in fact true for any regular surface locally at an non umbilical point.

Theorem Let $p \in M$ be a nonumbilical point of M. Then $\exists$ local coordinate patch $\mathbf{x}(u,v)$ around p such that the u- & v- parameter curves are lines of curvature.

That is, WLOG,

$$S(\mathbf{x}_u) = k_1 \mathbf{x}_u \quad \& \quad S(\mathbf{x}_v) = k_2 \mathbf{x}_v,$$

where S = shape operator, $k_1 > k_2$ are principal curvatures.
(non umbilical).

Pf: This theorem is an consequence of the theory of ODE.
The proof will be omitted.

Thm (H. Liebmann. Thm 3.5.5 of Oprea)

If M is a compact surface (in $\mathbb{R}^3$) of constant Gauss curvature K, then M is a sphere of radius $\frac{1}{\sqrt{K}}$.

To prove this thm, we need

Lemma (Hilbert. Lemma 3.5.6 of Oprea)

If $\exists$ $p \in M$ such that

$$k_1(p) = \max_M k_1 > k_2(p) = \min_M k_2,$$

then $K(p) \leq 0$.

We first prove the Liebmann thm by using Hilbert Lemma:

Pf of Liebmann Thm

M compact $\Rightarrow$ $\exists\ p_0 \in M$ s.t. $K(p_0) > 0$.

$\Rightarrow$ K is a positive constant.

Since $k_1(p)\,k_2(p) = K > 0$ constant & M cpt.,

$\exists\ p \in M$ s.t. $\begin{cases} k_1(p) = \max\limits_M k_1 \\ k_2(p) = \min\limits_M k_2 \end{cases}$ (where $k_1(p) \geq k_2(p)$ are the principal curvatures at p)

If $k_1(p) > k_2(p)$, then Hilbert lemma $\Rightarrow$ $K(p) \leq 0$

which is a contradiction.

$\therefore$ $k_1(p) = k_2(p)$

and hence $k_1 = k_2 = $ constant (on all surface M)

i.e. M is umbilic at all points.

Therefore (by thm 3.5.2 of Oprea) M contained in either a plane or a sphere. Since M cpt., M must be a sphere.

And the radius of the sphere is $\frac{1}{k_1} = \frac{1}{k_2} = \frac{1}{\sqrt{K}}$.

$\#$

Pf of Hilbert Lemma.

By continuity, $k_1 > k_2$ in a nbd. Ω of the point p.

Then by the thm. before the Liebmann theorem,

$\exists$ coordinate patch $X(u^1, u^2)$ in a possibly smaller nbd.

$\Omega_1 \subset \Omega$ of p such that

$$\begin{cases} S(X_1) = k_1 X_1 \\ S(X_2) = k_2 X_2. \end{cases}$$

Since $k_1 > k_2$, we have $\langle X_1, X_2 \rangle = 0$ (linear algebra using S symmetric)

i.e. $g_{12} = 0$

Then the 2^{nd} fundamental form (h_{ij}) are

$$\begin{cases} h_{11} = \langle S(X_1), X_1 \rangle = \langle k_1 X_1, X_1 \rangle = k_1 g_{11} \\ h_{12} = \langle S(X_1), X_2 \rangle = \langle k_1 X_1, X_2 \rangle = 0 \\ h_{22} = \langle S(X_2), X_2 \rangle = \langle k_2 X_2, X_2 \rangle = k_2 g_{22} \end{cases}$$

$\therefore$

$$(g_{ij}) = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} \;\&\; (h_{ij}) = \begin{pmatrix} k_1 g_{11} & 0 \\ 0 & k_2 g_{22} \end{pmatrix}$$

Recall the Codazzi - Mainardi equation

$$\frac{\partial h_{ij}}{\partial u^k} + \sum_{s=1}^{2} h_{ks} \Gamma_{ij}^s = \frac{\partial h_{ik}}{\partial u^j} + \sum_{s=1}^{2} h_{js} \Gamma_{ik}^s$$

Take $i = j = 1$ & $k = 2$, we have

$$\frac{\partial}{\partial u^2}(k_1 g_{11}) + h_{22} \Gamma_{11}^2 = h_{11} \Gamma_{12}^1 .$$

Now $g_{12} = 0 \Rightarrow \quad \Gamma_{11}^2 = -\frac{1}{2 g_{22}} \frac{\partial g_{11}}{\partial u^2}, \quad \Gamma_{12}^1 = \frac{1}{2 g_{11}} \frac{\partial g_{11}}{\partial u^2},$

$$\therefore \quad \frac{\partial}{\partial u^2}(k_1 g_{11}) = \frac{k_1}{2}\frac{\partial g_{11}}{\partial u^2} + \frac{k_2}{2}\frac{\partial g_{11}}{\partial u^2}$$

$$\Rightarrow \quad g_{11}\frac{\partial k_1}{\partial u^2} + k_1 \frac{\partial g_{11}}{\partial u^2} = \frac{k_1+k_2}{2}\frac{\partial g_{11}}{\partial u^2}$$

$$\Rightarrow \quad \boxed{\frac{\partial k_1}{\partial u^2} = \frac{(k_2-k_1)}{2 g_{11}} \cdot \frac{\partial g_{11}}{\partial u^2}}$$

If we take $i=j=2,\ k=1$ in

$$\frac{\partial h_{ij}}{\partial u^k} + \sum_{s=1}^{2} h_{ks}\Gamma^s_{ij} = \frac{\partial h_{ik}}{\partial u^j} + \sum_{s=1}^{2} h_{js}\Gamma^s_{ik},$$

we have $\dfrac{\partial}{\partial u^1}(k_2 g_{22}) + h_{11}\Gamma^1_{22} = h_{22}\Gamma^2_{21}$.

$$g_{12}=0 \Rightarrow \Gamma^1_{22} = \frac{-1}{2g_{11}}\frac{\partial g_{22}}{\partial u^1}, \quad \Gamma^2_{21} = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial u^1},$$

$$\Rightarrow \quad g_{22}\frac{\partial k_2}{\partial u^1} + k_2\frac{\partial g_{22}}{\partial u^1} = \frac{k_2}{2}\frac{\partial g_{22}}{\partial u^1} + \frac{k_1}{2}\frac{\partial g_{22}}{\partial u^1}$$

$$\Rightarrow \quad \boxed{\frac{\partial k_2}{\partial u^1} = \frac{(k_1 - k_2)}{2 g_{22}} \frac{\partial g_{22}}{\partial u^1}}$$

In conclusion, we have

$$(*)_1 \quad \begin{cases} \dfrac{\partial k_1}{\partial u^2} = -\dfrac{(k_1 - k_2)}{2 g_{11}} \dfrac{\partial g_{11}}{\partial u^2} \\[2ex] \dfrac{\partial k_2}{\partial u^1} = \dfrac{(k_1 - k_2)}{2 g_{22}} \dfrac{\partial g_{22}}{\partial u^1} \end{cases}$$

Differentiate again we have

$$(*)_2 \quad \begin{cases} \dfrac{\partial^2 k_1}{(\partial u^2)^2} = -\dfrac{(k_1 - k_2)}{2 g_{11}} \dfrac{\partial^2 g_{11}}{(\partial u^2)^2} + \left(\cdots \right) \dfrac{\partial g_{11}}{\partial u^2} \\[2ex] \dfrac{\partial^2 k_2}{(\partial u^1)^2} = \dfrac{k_1 - k_2}{2 g_{22}} \dfrac{\partial^2 g_{22}}{(\partial u^1)^2} + (\cdots) \dfrac{\partial g_{22}}{\partial u^1} \end{cases}$$

Then $k_1(p) = \max k_1$ & $k_2(p) = \min k_2$, we have

$$\frac{\partial k_1}{\partial u^2}(p) = 0 = \frac{\partial k_2}{\partial u^1}(p)$$

& $\dfrac{\partial^2 k_1}{(\partial u^2)^2}(p) \leq 0$ & $\dfrac{\partial^2 k_2}{(\partial u^1)^2}(p) \geq 0$

Putting into $(*)_1 \Rightarrow$ $\dfrac{\partial g_{11}}{\partial u^2}(p) = 0 = \dfrac{\partial g_{22}}{\partial u^1}(p)$

(Since $k_1 - k_2 > 0$ & $g_{11}(p) \neq 0$, $g_{22}(p) \neq 0$.)

and $\dfrac{\partial^2 g_{11}}{(\partial u^2)^2}(p) \geq 0$ & $\dfrac{\partial^2 g_{22}}{(\partial u^1)^2}(p) \geq 0$.

Finally, from Gauss equation

$$K = -\frac{1}{2\sqrt{g_{11}g_{22}}}\left[\frac{\partial}{\partial u^2}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{11}}{\partial u^2}\right) + \frac{\partial}{\partial u^1}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{22}}{\partial u^1}\right)\right]$$

$$= -\frac{1}{2(\sqrt{g_{11}g_{22}})^2}\left[\frac{\partial^2 g_{11}}{(\partial u^2)^2} + (\cdots)\frac{\partial g_{11}}{\partial u^2} + \frac{\partial^2 g_{22}}{(\partial u^1)^2} + (\cdots)\frac{\partial g_{22}}{\partial u^1}\right]$$

Hence

$$K = K(p) = -\frac{1}{2(\sqrt{g_{11}g_{22}})^2}\left[\frac{\partial^2 g_{11}}{(\partial u^2)^2}(p) + \frac{\partial^2 g_{22}}{(\partial u^1)^2}(p)\right] \leq 0.$$

$\divideontimes$

Thm (Thm 3.5.7. of Oprea)

If a surface of revolution M is minimal, then M is contained in either a plane or a catenoid.

Remark: A catenoid is the surface of revolution by revolving the catenary $y = \cosh(x)$ which up to change of parameter can be written as

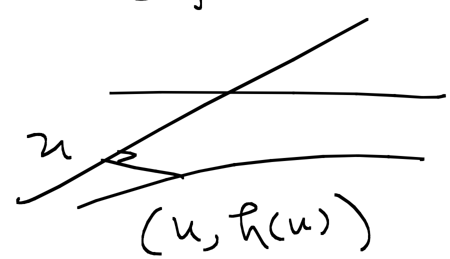
$$\alpha(t) = \left(t, \frac{1}{c}\cosh(ct \pm D)\right), \text{ where } c>0, D \text{ are constants}$$

Pf of the thm: A surface of revolution

$$\mathbf{x}(u,v) = (g(u), h(u)\cos v, h(u)\sin v)$$

can be reparametrized so that the parametrization takes the forms

$$\mathbf{x}(u,v) = (u, h(u)\cos v, h(u)\sin v)$$



near a point with $g'(u) \neq 0$.

If $g'(u) \equiv 0$, then $g = \text{const}$ and M contained in a plane.

Therefore, we only need to consider the case above.

Recall the following formula for the matrix representation of the shape operator

$$A = (g_{ij})^{-1}(h_{ij}) = \begin{pmatrix} \dfrac{g''h' - h''g'}{(g'^2 + h'^2)^{3/2}} & 0 \\ 0 & \dfrac{g'}{h\sqrt{g'^2 + h'^2}} \end{pmatrix}$$

$\Rightarrow$ mean curvature

$$H = \frac{1}{2}\left(\frac{-h''}{(1+h'^2)^{3/2}} + \frac{1}{h\sqrt{1+h'^2}}\right)$$

$$= \frac{-hh'' + 1 + h'^2}{2h(1+h'^2)^{3/2}}$$

$$\Rightarrow H \cdot 2h = \frac{1+h'^2 - hh''}{(1+h'^2)^{3/2}} = \frac{1}{(1+h'^2)^{1/2}} - \frac{hh''}{(1+h'^2)^{3/2}}$$

$$\Rightarrow H \cdot 2hh' = \frac{h'}{(1+h'^2)^{1/2}} - \frac{h\, h'h''}{(1+h'^2)^{3/2}}$$

$$\Rightarrow H[h^2]' = \left[\frac{h}{(1+h'^2)^{1/2}}\right]'$$

Minimal $\Rightarrow$ $H=0$ (by definition)

$$\Rightarrow \left[\frac{h}{(1+h'^2)^{1/2}}\right]' = 0$$

$$\Rightarrow \frac{h}{(1+h'^2)^{1/2}} = C \quad \text{constant}. \ (\Rightarrow C>0)$$

$$\Rightarrow c^2h^2 = 1+h'^2 \quad \text{where} \quad c = \frac{1}{C}$$

$$\Rightarrow h' = \pm\sqrt{c^2h^2-1}$$

$$\Rightarrow \quad \frac{dh}{\sqrt{c^2h^2-1}} = \pm du$$

Letting $ch = \cosh z$, we have

$$\frac{\frac{1}{c}\sinh z \, dz}{\sqrt{\cosh^2 z - 1}} = \pm du$$

$$\Rightarrow \quad dz = \pm c\,du$$

$$\Rightarrow \quad z = \pm cu + D$$

$$\therefore \quad ch = \cosh z = \cosh(cu \pm D)$$

$$\Rightarrow \quad h = \frac{1}{c}\cosh(cu \pm D).$$

This implies M is part of a catenoid. #

3.6 Surfaces of Delaunay (Just for reference, will not be included in final exam.)

(i.e. surface of revolution with constant mean curvature)

Thm (Thm 3.6.1 of Oprea)

A surface of revolution M parametrized by

$$\mathbf{x}(u,v) = (u,\ h(u)\cos v,\ h(u)\sin v)$$

has nonzero constant mean curvature $\iff$ $h(u)$ satisfies

$$h^2 \pm \frac{2ah}{\sqrt{1+h'^2}} = \pm b^2$$

where a & b are constants.

Pf: Recall that for the patch $\mathbf{x}(u,v) = (u,\ h(u)\cos v,\ h(u)\sin v)$

$$H\left[h^2\right]' = \left[\frac{h}{(1+h'^2)^{1/2}}\right]'$$

If H is a constant, then

$$H h^2 = \frac{h}{(1+h'^2)^{1/2}} + B \quad \text{for some constant } B$$

By assumption, H is a nonzero constant, we may write

$$H = \frac{1}{2a} \quad \text{for some constant } a \neq 0.$$

Then $$h^2 - \frac{2a\, h}{(1+h'^2)^{1/2}} = 2aB = \pm b^2 \quad \text{for some const. } b$$

Conversely, it is easy to see that h satisfies the equation with $a \neq 0$, then

$$2 h h' = 2a \left(\frac{1+h'^2 - h h''}{(1+h'^2)^{3/2}} \right) h'$$

Hence either $h'=0$ or $H = \frac{1}{2a}$.

If $h' = 0$, then $h =$ constant.

If $a = 0$, then $h^2 = \pm b^2 \implies h =$ constant.

In both cases, M is a cylinder with

$$H = \frac{1}{2h} \quad \text{a constant.} \qquad (h \neq 0 \text{ since } M \text{ is regular})$$

\#\#

(Ex. Read example 3.6.2 of Oprea)